

4次方程式の解の公式を求める

4次方程式 $ax^4 + bx^3 + cx^2 + dx + e = 0$ (a, b, c, d, e は実数定数、 $a \neq 0$) の解を求める。 $a \neq 0$ なので両辺を a で割って

$$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0$$

ここで三次の項を無くすために

$$y = \frac{b}{2^2a} + x \quad \Rightarrow \quad x = y - \frac{b}{2^2a} \quad (1)$$

と置く。 $(A-B)^4 = A^4 - 4A^3B + 6A^2B^2 - 4AB^3 + B^4$ 、 $(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$ なので、

$$\begin{aligned} \left(y^4 - \frac{b}{a}y^3 + \frac{3b^2}{2^3a^2}y^2 - \frac{b^3}{2^4a^3}y + \frac{b^4}{2^8a^4} \right) + \frac{b}{a} \left(y^3 - \frac{3b}{2^2a}y^2 + \frac{3b^2}{2^4a^2}y - \frac{b^3}{2^6a^3} \right) \\ + \frac{c}{a} \left(y^2 - \frac{b}{2a}y + \frac{b^2}{2^4a^2} \right) + \frac{d}{a} \left(y - \frac{b}{2^2a} \right) + \frac{e}{a} = 0 \end{aligned}$$

$$y^4 + \left(-\frac{3b^2}{2^3a^2} + \frac{c}{a} \right) y^2 + \left(\frac{b^3}{2^3a^3} - \frac{bc}{2a^2} + \frac{d}{a} \right) y + \left(-\frac{3b^4}{2^8a^4} + \frac{b^2c}{2^4a^3} - \frac{bd}{2^2a^2} + \frac{e}{a} \right) = 0$$

ここで

$$A_1 = -\frac{3b^2}{2^3a^2} + \frac{c}{a}, \quad A_2 = \frac{b^3}{2^3a^3} - \frac{bc}{2a^2} + \frac{d}{a}, \quad A_3 = -\frac{3b^4}{2^8a^4} + \frac{b^2c}{2^4a^3} - \frac{bd}{2^2a^2} + \frac{e}{a}$$

と置くと、

$$\begin{aligned} y^4 + A_1y^2 + A_2y + A_3 &= 0 \\ y^4 + A_1y^2 &= -A_2y - A_3 \end{aligned} \quad (2)$$

式 (2) の両辺に βy^2 を加える (β は現時点では未定の複素定数)

$$\begin{aligned} y^4 + (A_1 + \beta)y^2 &= \beta y^2 - A_2y - A_3 \\ \left\{ y^2 + \frac{1}{2}(A_1 + \beta) \right\}^2 - \frac{1}{4}(A_1 + \beta)^2 &= \beta \left(y - \frac{A_2}{2\beta} \right)^2 - \frac{A_2^2}{4\beta} - A_3 \\ \left\{ y^2 + \frac{1}{2}(A_1 + \beta) \right\}^2 &= \underbrace{\beta \left(y - \frac{A_2}{2\beta} \right)^2 - \frac{A_2^2}{4\beta} - A_3 + \frac{1}{4}(A_1 + \beta)^2}_{(A)} \end{aligned} \quad (3)$$

式 (3) の (A) の部分が 0 になるように β を決める。

$$\begin{aligned} -\frac{A_2^2}{4\beta} - A_3 + \frac{1}{4}(A_1^2 + 2A_1\beta + \beta^2) &= 0 \\ \beta^3 + 2A_1\beta^2 + (A_1^2 - 4A_3)\beta - A_2^2 &= 0 \end{aligned} \quad (4)$$

上記 β の三次方程式 (4) の解を $\beta_1, \beta_2, \beta_3$ とする。

(I) $\beta_1 = \beta_2 = \beta_3 = 0$ (解が0の三重根の場合)

ここでは式(3)の中で分母に β があることには目をつむる*1

$$A_1 = 0 \quad A_1^2 - 4A_3 = 0 \quad A_2 = 0 \quad (A_1 = 0 \text{ なので } 2 \text{ つ目の式は } A_3 = 0 \text{ と同等})$$

が成り立つ場合なので

$$A_1 = -\frac{3b^2}{2^3a^2} + \frac{c}{a} = 0 \quad -3b^2 + 2^3ac = 0 \quad (5)$$

$$A_2 = \frac{b^3}{2^3a^3} - \frac{bc}{2a^2} + \frac{d}{a} = 0 \quad b^3 - 2^2abc + 2^3a^2d = 0 \quad (6)$$

$$A_3 = -\frac{3b^4}{2^8a^4} + \frac{b^2c}{2^4a^3} - \frac{bd}{2^2a^2} + \frac{e}{a} = 0 \quad -3b^4 + 2^4ab^2c - 2^6a^2bd + 2^8a^3e = 0 \quad (7)$$

式(5)より $c = 3b^2/2^3a$ 、式(6)と式(7)より

$$d = \frac{1}{2^3a^2} (2^2abc - b^3) = \frac{1}{2^3a^2} \left(2^2ab \cdot \frac{3b^2}{2^3a} - b^3 \right) = \frac{1}{2^3a^2} \left(\frac{3b^3}{2} - b^3 \right) = \frac{b^3}{2^4a^2}$$

$$e = \frac{1}{2^8a^3} \left(3b^4 - 2^4ab^2c + 2^6a^2bd \right) = \frac{1}{2^8a^3} \left(3b^4 - \frac{2^4ab^2 \cdot 3b^2}{2 \cdot 3b^4} + \frac{2^6a^2b \cdot b^3}{2^2b^4} \right) = \frac{b^4}{2^8a^3}$$

よって、冒頭の四次方程式は

$$ax^4 + bx^3 + \frac{3b^2}{2^3a}x^2 + \frac{b^3}{2^4a^2}x + \frac{b^4}{2^8a^3} = 0$$

$$2^8a^4x^4 + 2^8a^3bx^3 + 2^53a^2b^2x^2 + 2^4ab^3x + b^4 = 0$$

$$(2^2ax + b)^4 = 0$$

$$\therefore x = -\frac{b}{2^2a} \quad (8)$$

(II) 解が0の三重根ではない場合

$A_2 = 0$ で解のひとつは0となるが、 $A_2 \neq 0$ の場合も含め、少なくとも β_1 、 β_2 、 β_3 のいずれかひとつは0ではない。 $\beta = \beta_1 \neq 0$ とすると、式(3)より

$$\left\{ y^2 + \frac{1}{2}(A_1 + \beta_1) \right\}^2 = \beta_1 \left(y - \frac{A_2}{2\beta_1} \right)^2$$

$$y^2 + \frac{1}{2}(A_1 + \beta_1) = \pm \sqrt{\beta_1} \left(y - \frac{A_2}{2\beta_1} \right)$$

$$y^2 \mp \sqrt{\beta_1}y + \frac{1}{2}A_1 + \frac{1}{2}\beta_1 \pm \frac{A_2}{2\sqrt{\beta_1}} = 0 \quad (\text{複号同順})$$

$$y = \frac{1}{2} \left\{ \pm \sqrt{\beta_1} \pm \sqrt{\beta_1 - 2A_1 - 2\beta_1 \mp \frac{2A_2}{\sqrt{\beta_1}}} \right\} = \frac{1}{2} \left\{ \begin{matrix} \text{(A)} \\ \pm \sqrt{\beta_1} \end{matrix} \begin{matrix} \text{(B)} \\ \pm \sqrt{-\beta_1 - 2A_1} \end{matrix} \begin{matrix} \text{(C)} \\ \mp \frac{2A_2}{\sqrt{\beta_1}} \end{matrix} \right\}$$

*1 $A_1 = A_2 = A_3 = 0$ の条件を式(3)より先に考えると、式(2)より $y = 0$ となるので、式(1)から式(8)の回答を得ることができるので、式(3)の分母に β があることは無関係となる。

ここで、(A) と (C) は複号同順、(B) の複号は任意。すなわち

$$y_a = \frac{1}{2} \left\{ \sqrt{\beta_1} \pm \sqrt{-\beta_1 - 2A_1 - \frac{2A_2}{\sqrt{\beta_1}}} \right\} \quad y_b = \frac{1}{2} \left\{ -\sqrt{\beta_1} \pm \sqrt{-\beta_1 - 2A_1 + \frac{2A_2}{\sqrt{\beta_1}}} \right\} \quad (9)$$

式 (4) に対する三次方程式の解と係数の関係から

$$\beta_1 \beta_2 \beta_3 = A_2^2 \quad (10)$$

$$\beta_1 + \beta_2 + \beta_3 = -2A_1 \quad (11)$$

式 (10) より $\sqrt{\beta_1} \sqrt{\beta_2} \sqrt{\beta_3} = \pm A_2$ 、式 (11) より $-\beta_1 - 2A_1 = \beta_2 + \beta_3$ なので、式 (9) の 2 つ目の根号の中は、

(i) $\sqrt{\beta_1} \sqrt{\beta_2} \sqrt{\beta_3} = -A_2$ ($A_2 = -\sqrt{\beta_1} \sqrt{\beta_2} \sqrt{\beta_3}$) となるように β_1 、 β_2 、 β_3 を決めると、
(i-a) y_a

$$-\beta_1 - 2A_1 - \frac{2A_2}{\sqrt{\beta_1}} = \beta_2 + \beta_3 + 2 \frac{\sqrt{\beta_1} \sqrt{\beta_2} \sqrt{\beta_3}}{\sqrt{\beta_1}} = \beta_2 + \beta_3 + 2\sqrt{\beta_2} \sqrt{\beta_3} = (\sqrt{\beta_2} + \sqrt{\beta_3})^2$$

よって、式 (9) 及び式 (1) は、

$$y_a = \frac{1}{2} \left(\sqrt{\beta_1} \pm \sqrt{(\sqrt{\beta_2} + \sqrt{\beta_3})^2} \right) = \frac{1}{2} \left(\sqrt{\beta_1} \pm \sqrt{\beta_2} \pm \sqrt{\beta_3} \right) \quad (\text{複号同順})$$

$$\therefore x_a = -\frac{b}{2^2 a} + \frac{1}{2} \left(\sqrt{\beta_1} \pm \sqrt{\beta_2} \pm \sqrt{\beta_3} \right) \quad (\text{複号同順})$$

(i-b) y_b

$$-\beta_1 - 2A_1 + \frac{2A_2}{\sqrt{\beta_1}} = \beta_2 + \beta_3 - 2 \frac{\sqrt{\beta_1} \sqrt{\beta_2} \sqrt{\beta_3}}{\sqrt{\beta_1}} = \beta_2 + \beta_3 - 2\sqrt{\beta_2} \sqrt{\beta_3} = (\sqrt{\beta_2} - \sqrt{\beta_3})^2$$

よって、式 (9) 及び式 (1) は、

$$y_b = \frac{1}{2} \left(-\sqrt{\beta_1} \pm \sqrt{(\sqrt{\beta_2} - \sqrt{\beta_3})^2} \right) = \frac{1}{2} \left(-\sqrt{\beta_1} \pm \sqrt{\beta_2} \mp \sqrt{\beta_3} \right) \quad (\text{複号同順})$$

$$\therefore x_b = -\frac{b}{2^2 a} + \frac{1}{2} \left(-\sqrt{\beta_1} \pm \sqrt{\beta_2} \mp \sqrt{\beta_3} \right) \quad (\text{複号同順})$$

(i-a)、(i-b) の結果をまとめると、

$$\left. \begin{aligned} x_1 &= -\frac{b}{2^2 a} + \frac{1}{2} \left(\sqrt{\beta_1} + \sqrt{\beta_2} + \sqrt{\beta_3} \right) \\ x_2 &= -\frac{b}{2^2 a} + \frac{1}{2} \left(\sqrt{\beta_1} - \sqrt{\beta_2} - \sqrt{\beta_3} \right) \\ x_3 &= -\frac{b}{2^2 a} + \frac{1}{2} \left(-\sqrt{\beta_1} + \sqrt{\beta_2} - \sqrt{\beta_3} \right) \\ x_4 &= -\frac{b}{2^2 a} + \frac{1}{2} \left(-\sqrt{\beta_1} - \sqrt{\beta_2} + \sqrt{\beta_3} \right) \end{aligned} \right\} \begin{aligned} x_i &= -\frac{b}{2^2 a} + \frac{1}{2} \left(l\sqrt{\beta_1} + m\sqrt{\beta_2} + n\sqrt{\beta_3} \right) \\ (l, m, n) &= (1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1) \\ \beta_1 = \beta_2 = \beta_3 = 0 &\text{ の解 (8) もこの式で表現されている} \end{aligned}$$

$$\text{但し、} \sqrt{\beta_1} \sqrt{\beta_2} \sqrt{\beta_3} = -A_2 = -\frac{b^3}{2^3 a^3} + \frac{bc}{2a^2} - \frac{d}{a}$$

(ii) $\sqrt{\beta_1}\sqrt{\beta_2}\sqrt{\beta_3} = A_2$ ($A_2 = \sqrt{\beta_1}\sqrt{\beta_2}\sqrt{\beta_3}$) となるように $\beta_1, \beta_2, \beta_3$ を決めると、

(ii-a) y_a

$$-\beta_1 - 2A_1 - \frac{2A_2}{\sqrt{\beta_1}} = \beta_2 + \beta_3 - 2\frac{\sqrt{\beta_1}\sqrt{\beta_2}\sqrt{\beta_3}}{\sqrt{\beta_1}} = \beta_2 + \beta_3 - 2\sqrt{\beta_2}\sqrt{\beta_3} = (\sqrt{\beta_2} - \sqrt{\beta_3})^2$$

よって、式 (9) 及び式 (1) は、

$$y_a = \frac{1}{2}(\sqrt{\beta_1} \pm \sqrt{\beta_2} \mp \sqrt{\beta_3}) \quad \therefore x_a = -\frac{b}{2^2a} + \frac{1}{2}(\sqrt{\beta_1} \pm \sqrt{\beta_2} \mp \sqrt{\beta_3}) \quad (\text{複号同順})$$

(ii-b) y_b

$$-\beta_1 - 2A_1 + \frac{2A_2}{\sqrt{\beta_1}} = \beta_2 + \beta_3 + 2\frac{\sqrt{\beta_1}\sqrt{\beta_2}\sqrt{\beta_3}}{\sqrt{\beta_1}} = \beta_2 + \beta_3 + 2\sqrt{\beta_2}\sqrt{\beta_3} = (\sqrt{\beta_2} + \sqrt{\beta_3})^2$$

よって、式 (9) 及び式 (1) は、

$$y_b = \frac{1}{2}(-\sqrt{\beta_1} \pm \sqrt{\beta_2} \pm \sqrt{\beta_3}) \quad \therefore x_b = -\frac{b}{2^2a} + \frac{1}{2}(-\sqrt{\beta_1} \pm \sqrt{\beta_2} \pm \sqrt{\beta_3}) \quad (\text{複号同順})$$

(ii-a)、(ii-b) の結果をまとめると、

$$\left. \begin{aligned} x_1 &= -\frac{b}{2^2a} + \frac{1}{2}(\sqrt{\beta_1} + \sqrt{\beta_2} - \sqrt{\beta_3}) \\ x_2 &= -\frac{b}{2^2a} + \frac{1}{2}(\sqrt{\beta_1} - \sqrt{\beta_2} + \sqrt{\beta_3}) \\ x_3 &= -\frac{b}{2^2a} + \frac{1}{2}(-\sqrt{\beta_1} + \sqrt{\beta_2} + \sqrt{\beta_3}) \\ x_4 &= -\frac{b}{2^2a} + \frac{1}{2}(-\sqrt{\beta_1} - \sqrt{\beta_2} - \sqrt{\beta_3}) \end{aligned} \right\} \begin{aligned} x_i &= -\frac{b}{2^2a} + \frac{1}{2}(l\sqrt{\beta_1} + m\sqrt{\beta_2} + n\sqrt{\beta_3}) \\ (l, m, n) &= (1, 1, -1), (1, -1, 1), (-1, 1, 1), (-1, -1, -1) \\ \beta_1 = \beta_2 = \beta_3 = 0 &\text{ の解 (8) もこの式で表現されている} \end{aligned}$$

$$\text{但し、}\sqrt{\beta_1}\sqrt{\beta_2}\sqrt{\beta_3} = A_2 = \frac{b^3}{2^3a^3} - \frac{bc}{2a^2} + \frac{d}{a}$$

以上をまとめると $ax^4 + bx^3 + cx^2 + dx + e = 0$ ($a \neq 0$) の解は、

$$x_i = -\frac{b}{2^2a} + \frac{l\sqrt{\beta_1} + m\sqrt{\beta_2} + n\sqrt{\beta_3}}{2}$$

$$(l, m, n) = (1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)$$

$$\text{但し、}\sqrt{\beta_1}\sqrt{\beta_2}\sqrt{\beta_3} = -\frac{b^3}{2^3a^3} + \frac{bc}{2a^2} - \frac{d}{a}$$

及び、

$$x_i = -\frac{b}{2^2a} + \frac{l\sqrt{\beta_1} + m\sqrt{\beta_2} + n\sqrt{\beta_3}}{2}$$

$$(l, m, n) = (1, 1, -1), (1, -1, 1), (-1, 1, 1), (-1, -1, -1)$$

$$\text{但し、}\sqrt{\beta_1}\sqrt{\beta_2}\sqrt{\beta_3} = \frac{b^3}{2^3a^3} - \frac{bc}{2a^2} + \frac{d}{a}$$

の 2 種類存在し、これらは同じ結果となる。

β_i は式 (4) の三次方程式の解なので、三次方程式の解の公式を当てはめて実際に求めてみる。

$$\beta^3 + 2A_1\beta^2 + (A_1^2 - 4A_3)\beta - A_2^2 = 0 \quad (a = 1, b = 2A_1, c = A_1^2 - 4A_3, d = -A_2^2)$$

三次方程式の解の公式に現れる記号を $X \rightarrow U, Y \rightarrow V, P \rightarrow S, Q \rightarrow T$ と置き換えて、

$$\begin{aligned} U &= -2^2 3^3 (-A_2^2) + 2^2 3^2 \cdot 2A_1 (A_1^2 - 2^2 A_3) - 2^3 (2A_1)^3 \\ &= 2^2 3^3 A_2^2 + 2^3 3^2 A_1^3 - 2^5 3^2 A_1 A_3 - 2^6 A_1^3 \\ \therefore U &= 2^3 A_1^3 - 2^5 3^2 A_1 A_3 + 2^2 3^3 A_2^2 \end{aligned} \tag{12}$$

$$\begin{aligned} V &= 3^4 (-A_2^2)^2 - 2 \cdot 3^3 \cdot 2A_1 (A_1^2 - 2^2 A_3) (-A_2^2) + 2^2 3 (A_1^2 - 2^2 A_3)^3 + 2^2 3 \cdot 2^3 A_1^3 (-A_2^2) - 3 \cdot 2^2 A_1^2 (A_1^2 - 2^2 A_3)^2 \\ &= 3^4 A_2^4 + 2^2 3^3 A_1^3 A_2^2 - 2^4 3^3 A_1 A_2^2 A_3 \\ &\quad + 2^2 3 (A_1^6 - 2^2 3 A_1^4 A_3 + 2^4 3 A_1^2 A_3^2 - 2^6 A_3^3) - 2^5 3 A_1^3 A_2^2 - 2^2 3 A_1^2 (A_1^4 - 2^3 A_1^2 A_3 + 2^4 A_3^2) \\ &= 3^4 A_2^4 + 2^2 3^3 A_1^3 A_2^2 - 2^4 3^3 A_1 A_2^2 A_3 + 2^2 3 A_1^6 - 2^4 3^2 A_1^4 A_3 \\ &\quad + 2^6 3^2 A_1^2 A_3^2 - 2^8 3 A_3^3 - 2^5 3 A_1^3 A_2^2 - 2^2 3 A_1^6 + 2^5 3 A_1^4 A_3 - 2^6 3 A_1^2 A_3^2 \\ \therefore V &= -2^4 3 A_1^4 A_3 + 2^2 3 A_1^3 A_2^2 + 2^7 3 A_1^2 A_3^2 - 2^4 3^3 A_1 A_2^2 A_3 + 3^4 A_2^4 - 2^8 3 A_3^3 \quad \dots\dots\dots (A) = 0 \end{aligned} \tag{13}$$

$$S = U + 2^2 \cdot 3\sqrt{V} \tag{14}$$

$$T = U - 2^2 \cdot 3\sqrt{V} \tag{15}$$

として、

$$\begin{aligned} \beta_i &= \frac{1}{2 \cdot 3} \left(-2^2 A_1 + s S^{\frac{1}{3}} + t T^{\frac{1}{3}} \right) = -\frac{2}{3} A_1 + \frac{s}{2 \cdot 3} S^{\frac{1}{3}} + \frac{t}{2 \cdot 3} T^{\frac{1}{3}} = -\frac{2}{3} \left(-\frac{3b^2}{2^3 a^2} + \frac{c}{a} \right) + \frac{s}{2 \cdot 3} S^{\frac{1}{3}} + \frac{t}{2 \cdot 3} T^{\frac{1}{3}} \\ \therefore \beta_i &= \frac{b^2}{2^2 a^2} - \frac{2c}{3a} + \frac{s}{2 \cdot 3} S^{\frac{1}{3}} + \frac{t}{2 \cdot 3} T^{\frac{1}{3}} \end{aligned} \tag{16}$$

但し、 S と T は複素数で、次の式を満たすものの内一組を選ぶ。

$$S^{\frac{1}{3}} T^{\frac{1}{3}} = 2^2 (2A_1)^2 - 2^2 3 (A_1^2 - 2^2 A_3) = 2^4 A_1^2 - 2^2 3 A_1^2 + 2^4 \cdot 3 A_3 = 2^2 A_1^2 + 2^4 3 A_3 \tag{17}$$

また、 (s, t) は次の三組のいずれかの値を取るものとし、 ω, ω^2 は 1 の三乗根の複素解とする。

$$(s, t) = (1, 1), (\omega, \omega^2), (\omega^2, \omega) \quad \omega = \frac{-1 + \sqrt{3}i}{2}, \quad \omega^2 = \frac{-1 - \sqrt{3}i}{2} \quad (\omega^3 = 1, \omega + \omega^2 = -1)$$

$$\begin{aligned} A_1^2 &= \left(-\frac{3b^2}{2^3 a^2} + \frac{c}{a} \right)^2 = \frac{3^2 b^4}{2^6 a^4} - \frac{3b^2 c}{2^2 a^3} + \frac{c^2}{a^2} \\ A_1^3 &= \left(-\frac{3b^2}{2^3 a^2} + \frac{c}{a} \right) \left(\frac{3^2 b^4}{2^6 a^4} - \frac{3b^2 c}{2^2 a^3} + \frac{c^2}{a^2} \right) = -\frac{3^3 b^6}{2^9 a^6} + \frac{3^2 b^4 c}{2^5 a^5} - \frac{3b^2 c^2}{2^3 a^4} + \frac{3^2 b^4 c}{2^6 a^5} - \frac{3b^2 c^2}{2^2 a^4} + \frac{c^3}{a^3} \\ &= -\frac{3^3 b^6}{2^9 a^6} + \frac{3^3 b^4 c}{2^6 a^5} - \frac{3^2 b^2 c^2}{2^3 a^4} + \frac{c^3}{a^3} \end{aligned}$$

$$\begin{aligned}
 A_1^4 &= \left(\frac{3^2 b^4}{2^6 a^4} - \frac{3b^2 c}{2^2 a^3} + \frac{c^2}{a^2} \right) \left(\frac{3^2 b^4}{2^6 a^4} - \frac{3b^2 c}{2^2 a^3} + \frac{c^2}{a^2} \right) \\
 &= \frac{3^4 b^8}{2^{12} a^8} - \frac{3^3 b^6 c}{2^8 a^7} + \frac{3^2 b^4 c^2}{2^6 a^6} - \frac{3^3 b^6 c}{2^8 a^7} + \frac{3^2 b^4 c^2}{2^4 a^6} - \frac{3b^2 c^3}{2^2 a^5} + \frac{3^2 b^4 c^2}{2^6 a^6} - \frac{3b^2 c^3}{2^2 a^5} + \frac{c^4}{a^4} \\
 &= \frac{3^4 b^8}{2^{12} a^8} - \frac{3^3 b^6 c}{2^7 a^7} + \frac{3^3 b^4 c^2}{2^5 a^6} - \frac{3b^2 c^3}{2a^5} + \frac{c^4}{a^4}
 \end{aligned}$$

$$\begin{aligned}
 A_2^2 &= \left(\frac{b^3}{2^3 a^3} - \frac{bc}{2a^2} + \frac{d}{a} \right) \left(\frac{b^3}{2^3 a^3} - \frac{bc}{2a^2} + \frac{d}{a} \right) \\
 &= \frac{b^6}{2^6 a^6} - \frac{b^4 c}{2^4 a^5} + \frac{b^3 d}{2^3 a^4} - \frac{b^4 c}{2^4 a^5} + \frac{b^2 c^2}{2^2 a^4} - \frac{bcd}{2a^3} + \frac{b^3 d}{2^3 a^4} - \frac{bcd}{2a^3} + \frac{d^2}{a^2} \\
 &= \frac{b^6}{2^6 a^6} - \frac{b^4 c}{2^3 a^5} + \frac{b^3 d}{2^2 a^4} + \frac{b^2 c^2}{2^2 a^4} - \frac{bcd}{a^3} + \frac{d^2}{a^2}
 \end{aligned}$$

$$\begin{aligned}
 A_2^4 &= \left(\frac{b^6}{2^6 a^6} - \frac{b^4 c}{2^3 a^5} + \frac{b^3 d}{2^2 a^4} + \frac{b^2 c^2}{2^2 a^4} - \frac{bcd}{a^3} + \frac{d^2}{a^2} \right) \left(\frac{b^6}{2^6 a^6} - \frac{b^4 c}{2^3 a^5} + \frac{b^3 d}{2^2 a^4} + \frac{b^2 c^2}{2^2 a^4} - \frac{bcd}{a^3} + \frac{d^2}{a^2} \right) \\
 &= \frac{b^{12}}{2^{12} a^{12}} - \frac{b^{10} c}{2^9 a^{11}} + \frac{b^9 d}{2^8 a^{10}} + \frac{b^8 c^2}{2^8 a^{10}} - \frac{b^7 cd}{2^6 a^9} + \frac{b^6 d^2}{2^6 a^8} - \frac{b^{10} c}{2^9 a^{11}} + \frac{b^8 c^2}{2^6 a^{10}} - \frac{b^7 cd}{2^5 a^9} - \frac{b^6 c^3}{2^5 a^9} + \frac{b^5 c^2 d}{2^3 a^8} - \frac{b^4 cd^2}{2^3 a^7} \\
 &\quad + \frac{b^9 d}{2^8 a^{10}} - \frac{b^7 cd}{2^5 a^9} + \frac{b^6 d^2}{2^4 a^8} + \frac{b^5 c^2 d}{2^4 a^8} - \frac{b^4 cd^2}{2^2 a^7} + \frac{b^3 d^3}{2^2 a^6} + \frac{b^8 c^2}{2^8 a^{10}} - \frac{b^6 c^3}{2^5 a^9} + \frac{b^5 c^2 d}{2^4 a^8} + \frac{b^4 c^4}{2^4 a^8} - \frac{b^3 c^3 d}{2^2 a^7} + \frac{b^2 c^2 d^2}{2^2 a^6} \\
 &\quad - \frac{b^7 cd}{2^6 a^9} + \frac{b^5 c^2 d}{2^3 a^8} - \frac{b^4 cd^2}{2^2 a^7} - \frac{b^3 c^3 d}{2^2 a^7} + \frac{b^2 c^2 d^2}{a^6} - \frac{bcd^3}{a^5} + \frac{b^6 d^2}{2^6 a^8} - \frac{b^4 cd^2}{2^3 a^7} + \frac{b^3 d^3}{2^2 a^6} + \frac{b^2 c^2 d^2}{2^2 a^6} - \frac{bcd^3}{a^5} + \frac{d^4}{a^4} \\
 &= \frac{b^{12}}{2^{12} a^{12}} - \frac{b^{10} c}{2^8 a^{11}} + \frac{b^9 d}{2^7 a^{10}} + \frac{3b^8 c^2}{2^7 a^{10}} - \frac{3b^7 cd}{2^5 a^9} - \frac{b^6 c^3}{2^4 a^9} + \frac{3b^6 d^2}{2^5 a^8} + \frac{3b^5 c^2 d}{2^3 a^8} + \frac{b^4 c^4}{2^4 a^8} \\
 &\quad - \frac{3b^4 cd^2}{2^2 a^7} - \frac{b^3 c^3 d}{2a^7} + \frac{b^3 d^3}{2a^6} + \frac{3b^2 c^2 d^2}{2a^6} - \frac{2bcd^3}{a^5} + \frac{d^4}{a^4}
 \end{aligned}$$

$$\begin{aligned}
 A_3^2 &= \left(-\frac{3b^4}{2^8 a^4} + \frac{b^2 c}{2^4 a^3} - \frac{bd}{2^2 a^2} + \frac{e}{a} \right) \left(-\frac{3b^4}{2^8 a^4} + \frac{b^2 c}{2^4 a^3} - \frac{bd}{2^2 a^2} + \frac{e}{a} \right) \\
 &= \frac{3^2 b^8}{2^{16} a^8} - \frac{3b^6 c}{2^{12} a^7} + \frac{3b^5 d}{2^{10} a^6} - \frac{3b^4 e}{2^8 a^5} - \frac{3b^6 c}{2^{12} a^7} + \frac{b^4 c^2}{2^8 a^6} - \frac{b^3 cd}{2^6 a^5} + \frac{b^2 ce}{2^4 a^4} \\
 &\quad + \frac{3b^5 d}{2^{10} a^6} - \frac{b^3 cd}{2^6 a^5} + \frac{b^2 d^2}{2^4 a^4} - \frac{bde}{2^2 a^3} - \frac{3b^4 e}{2^8 a^5} + \frac{b^2 ce}{2^4 a^4} - \frac{bde}{2^2 a^3} + \frac{e^2}{a^2} \\
 &= \frac{3^2 b^8}{2^{16} a^8} - \frac{3b^6 c}{2^{11} a^7} + \frac{3b^5 d}{2^9 a^6} + \frac{b^4 c^2}{2^8 a^6} - \frac{3b^4 e}{2^7 a^5} - \frac{b^3 cd}{2^5 a^5} + \frac{b^2 ce}{2^3 a^4} + \frac{b^2 d^2}{2^4 a^4} - \frac{bde}{2a^3} + \frac{e^2}{a^2}
 \end{aligned}$$

$$\begin{aligned}
 A_3^3 &= \left(-\frac{3b^4}{2^8a^4} + \frac{b^2c}{2^4a^3} - \frac{bd}{2^2a^2} + \frac{e}{a} \right) \left(\frac{3^2b^8}{2^{16}a^8} - \frac{3b^6c}{2^{11}a^7} + \frac{3b^5d}{2^9a^6} + \frac{b^4c^2}{2^8a^6} - \frac{3b^4e}{2^7a^5} - \frac{b^3cd}{2^5a^5} + \frac{b^2ce}{2^3a^4} + \frac{b^2d^2}{2^4a^4} - \frac{bde}{2a^3} + \frac{e^2}{a^2} \right) \\
 &= -\frac{3^3b^{12}}{2^{24}a^{12}} + \frac{3^2b^{10}c}{2^{19}a^{11}} - \frac{3^2b^9d}{2^{17}a^{10}} - \frac{3b^8c^2}{2^{16}a^{10}} + \frac{3^2b^8e}{2^{15}a^9} + \frac{3b^7cd}{2^{13}a^9} - \frac{3b^6ce}{2^{11}a^8} - \frac{3b^6d^2}{2^{12}a^8} + \frac{3b^5de}{2^9a^7} - \frac{3b^4e^2}{2^8a^6} \\
 &\quad + \frac{3^2b^{10}c}{2^{20}a^{11}} - \frac{3b^8c^2}{2^{15}a^{10}} + \frac{3b^7cd}{2^{13}a^9} + \frac{b^6c^3}{2^{12}a^9} - \frac{3b^6ce}{2^{11}a^8} - \frac{b^5c^2d}{2^9a^8} + \frac{b^4c^2e}{2^7a^7} + \frac{b^4cd^2}{2^8a^7} - \frac{b^3cde}{2^5a^6} + \frac{b^2ce^2}{2^4a^5} \\
 &\quad - \frac{3^2b^9d}{2^{18}a^{10}} + \frac{3b^7cd}{2^{13}a^9} - \frac{3b^6d^2}{2^{11}a^8} - \frac{b^5c^2d}{2^{10}a^8} + \frac{3b^5de}{2^9a^7} + \frac{b^4cd^2}{2^7a^7} - \frac{b^3cde}{2^5a^6} - \frac{b^3d^3}{2^6a^6} + \frac{b^2d^2e}{2^3a^5} - \frac{bde^2}{2^2a^4} \\
 &\quad + \frac{3^2b^8e}{2^{16}a^9} - \frac{3b^6ce}{2^{11}a^8} + \frac{3b^5de}{2^9a^7} + \frac{b^4c^2e}{2^8a^7} - \frac{3b^4e^2}{2^7a^6} - \frac{b^3cde}{2^5a^6} + \frac{b^2ce^2}{2^3a^5} + \frac{b^2d^2e}{2^4a^5} - \frac{bde^2}{2a^4} + \frac{e^3}{a^3} \\
 &= -\frac{3^3b^{12}}{2^{24}a^{12}} + \frac{3^3b^{10}c}{2^{20}a^{11}} - \frac{3^3b^9d}{2^{18}a^{10}} - \frac{3^2b^8c^2}{2^{16}a^{10}} + \frac{3^3b^8e}{2^{16}a^9} + \frac{3^2b^7cd}{2^{13}a^9} + \frac{b^6c^3}{2^{12}a^9} - \frac{3^2b^6ce}{2^{11}a^8} - \frac{3^2b^6d^2}{2^{12}a^8} - \frac{3b^5c^2d}{2^{10}a^8} \\
 &\quad + \frac{3^2b^5de}{2^9a^7} + \frac{3b^4c^2e}{2^8a^7} + \frac{3b^4cd^2}{2^8a^7} - \frac{3^2b^4e^2}{2^8a^6} - \frac{3b^3cde}{2^5a^6} - \frac{b^3d^3}{2^6a^6} + \frac{3b^2ce^2}{2^4a^5} + \frac{3b^2d^2e}{2^4a^5} - \frac{3bde^2}{2^2a^4} + \frac{e^3}{a^3}
 \end{aligned}$$

$$\begin{aligned}
 A_1A_3 &= \left(-\frac{3b^2}{2^3a^2} + \frac{c}{a} \right) \left(-\frac{3b^4}{2^8a^4} + \frac{b^2c}{2^4a^3} - \frac{bd}{2^2a^2} + \frac{e}{a} \right) \\
 &= \frac{3^2b^6}{2^{11}a^6} - \frac{3b^4c}{2^7a^5} + \frac{3b^3d}{2^5a^4} - \frac{3b^2e}{2^3a^3} - \frac{3b^4c}{2^8a^5} + \frac{b^2c^2}{2^4a^4} - \frac{bcd}{2^2a^3} + \frac{ce}{a^2} \\
 &= \frac{3^2b^6}{2^{11}a^6} - \frac{3^2b^4c}{2^8a^5} + \frac{3b^3d}{2^5a^4} + \frac{b^2c^2}{2^4a^4} - \frac{3b^2e}{2^3a^3} - \frac{bcd}{2^2a^3} + \frac{ce}{a^2}
 \end{aligned}$$

$$\begin{aligned}
 A_1^4A_3 &= \left(\frac{3^4b^8}{2^{12}a^8} - \frac{3^3b^6c}{2^7a^7} + \frac{3^3b^4c^2}{2^5a^6} - \frac{3b^2c^3}{2a^5} + \frac{c^4}{a^4} \right) \left(-\frac{3b^4}{2^8a^4} + \frac{b^2c}{2^4a^3} - \frac{bd}{2^2a^2} + \frac{e}{a} \right) \\
 &= -\frac{3^5b^{12}}{2^{20}a^{12}} + \frac{3^4b^{10}c}{2^{16}a^{11}} - \frac{3^4b^9d}{2^{14}a^{10}} + \frac{3^4b^8e}{2^{12}a^9} + \frac{3^4b^{10}c}{2^{15}a^{11}} - \frac{3^3b^8c^2}{2^{11}a^{10}} + \frac{3^3b^7cd}{2^9a^9} - \frac{3^3b^6ce}{2^7a^8} \\
 &\quad - \frac{3^4b^8c^2}{2^{13}a^{10}} + \frac{3^3b^6c^3}{2^9a^9} - \frac{3^3b^5c^2d}{2^7a^8} + \frac{3^3b^4c^2e}{2^5a^7} + \frac{3^2b^6c^3}{2^9a^9} - \frac{3b^4c^4}{2^5a^8} + \frac{3b^3c^3d}{2^3a^7} - \frac{3b^2c^3e}{2a^6} \\
 &\quad - \frac{3b^4c^4}{2^8a^8} + \frac{b^2c^5}{2^4a^7} - \frac{bc^4d}{2^2a^6} + \frac{c^4e}{a^5} \\
 &= -\frac{3^5b^{12}}{2^{20}a^{12}} + \frac{3^5b^{10}c}{2^{16}a^{11}} - \frac{3^4b^9d}{2^{14}a^{10}} - \frac{3^3b^8c^2}{2^{13}a^{10}} + \frac{3^4b^8e}{2^{12}a^9} + \frac{3^3b^7cd}{2^9a^9} + \frac{3^2b^6c^3}{2^7a^9} \\
 &\quad - \frac{3^3b^6ce}{2^7a^8} - \frac{3^3b^5c^2d}{2^7a^8} - \frac{3^3b^4c^4}{2^8a^8} + \frac{3^3b^4c^2e}{2^5a^7} + \frac{3b^3c^3d}{2^3a^7} + \frac{b^2c^5}{2^4a^7} - \frac{3b^2c^3e}{2a^6} - \frac{bc^4d}{2^2a^6} + \frac{c^4e}{a^5}
 \end{aligned}$$

$$\begin{aligned}
 A_1^3 A_2^2 &= \left(-\frac{3^3 b^6}{2^9 a^6} + \frac{3^3 b^4 c}{2^6 a^5} - \frac{3^2 b^2 c^2}{2^3 a^4} + \frac{c^3}{a^3} \right) \left(\frac{b^6}{2^6 a^6} - \frac{b^4 c}{2^3 a^5} + \frac{b^3 d}{2^2 a^4} + \frac{b^2 c^2}{2^2 a^4} - \frac{bcd}{a^3} + \frac{d^2}{a^2} \right) \\
 &= -\frac{3^3 b^{12}}{2^{15} a^{12}} + \frac{3^3 b^{10} c}{2^{12} a^{11}} - \frac{3^3 b^9 d}{2^{11} a^{10}} - \frac{3^3 b^8 c^2}{2^{11} a^{10}} + \frac{3^3 b^7 cd}{2^9 a^9} - \frac{3^3 b^6 d^2}{2^9 a^8} \\
 &\quad + \frac{3^3 b^{10} c}{2^{12} a^{11}} - \frac{3^3 b^8 c^2}{2^9 a^{10}} + \frac{3^3 b^7 cd}{2^8 a^9} + \frac{3^3 b^6 c^3}{2^8 a^9} - \frac{3^3 b^5 c^2 d}{2^6 a^8} + \frac{3^3 b^4 cd^2}{2^6 a^7} \\
 &\quad - \frac{3^2 b^8 c^2}{2^9 a^{10}} + \frac{3^2 b^6 c^3}{2^6 a^9} - \frac{3^2 b^5 c^2 d}{2^5 a^8} - \frac{3^2 b^4 c^4}{2^5 a^8} + \frac{3^2 b^3 c^3 d}{2^3 a^7} - \frac{3^2 b^2 c^2 d^2}{2^3 a^6} \\
 &\quad + \frac{b^6 c^3}{2^6 a^9} - \frac{b^4 c^4}{2^3 a^8} + \frac{b^3 c^3 d}{2^2 a^7} + \frac{b^2 c^5}{2^2 a^7} - \frac{bc^4 d}{a^6} + \frac{c^3 d^2}{a^5} \\
 &= -\frac{3^3 b^{12}}{2^{15} a^{12}} + \frac{3^3 b^{10} c}{2^{11} a^{11}} - \frac{3^3 b^9 d}{2^{11} a^{10}} - \frac{3^2 19 b^8 c^2}{2^{11} a^{10}} + \frac{3^4 b^7 cd}{2^9 a^9} + \frac{67 b^6 c^3}{2^8 a^9} - \frac{3^3 b^6 d^2}{2^9 a^8} - \frac{3^2 5 b^5 c^2 d}{2^6 a^8} - \frac{13 b^4 c^4}{2^5 a^8} \\
 &\quad + \frac{3^3 b^4 cd^2}{2^6 a^7} + \frac{11 b^3 c^3 d}{2^3 a^7} + \frac{b^2 c^5}{2^2 a^7} - \frac{3^2 b^2 c^2 d^2}{2^3 a^6} - \frac{bc^4 d}{a^6} + \frac{c^3 d^2}{a^5}
 \end{aligned}$$

$$\begin{aligned}
 A_1^2 A_3^2 &= \left(\frac{3^2 b^4}{2^6 a^4} - \frac{3b^2 c}{2^2 a^3} + \frac{c^2}{a^2} \right) \left(\frac{3^2 b^8}{2^{16} a^8} - \frac{3b^6 c}{2^{11} a^7} + \frac{3b^5 d}{2^9 a^6} + \frac{b^4 c^2}{2^8 a^6} - \frac{3b^4 e}{2^7 a^5} - \frac{b^3 cd}{2^5 a^5} + \frac{b^2 ce}{2^3 a^4} + \frac{b^2 d^2}{2^4 a^4} - \frac{bde}{2a^3} + \frac{e^2}{a^2} \right) \\
 &= \frac{3^4 b^{12}}{2^{22} a^{12}} - \frac{3^3 b^{10} c}{2^{17} a^{11}} + \frac{3^3 b^9 d}{2^{15} a^{10}} + \frac{3^2 b^8 c^2}{2^{14} a^{10}} - \frac{3^3 b^8 e}{2^{13} a^9} - \frac{3^2 b^7 cd}{2^{11} a^9} + \frac{3^2 b^6 ce}{2^9 a^8} + \frac{3^2 b^6 d^2}{2^{10} a^8} - \frac{3^2 b^5 de}{2^7 a^7} + \frac{3^2 b^4 e^2}{2^6 a^6} \\
 &\quad - \frac{3^3 b^{10} c}{2^{18} a^{11}} + \frac{3^2 b^8 c^2}{2^{13} a^{10}} - \frac{3^2 b^7 cd}{2^{11} a^9} - \frac{3b^6 c^3}{2^{10} a^9} + \frac{3^2 b^6 ce}{2^9 a^8} + \frac{3b^5 c^2 d}{2^7 a^8} - \frac{3b^4 c^2 e}{2^5 a^7} - \frac{3b^4 cd^2}{2^6 a^7} + \frac{3b^3 cde}{2^3 a^6} - \frac{3b^2 ce^2}{2^2 a^5} \\
 &\quad + \frac{3^2 b^8 c^2}{2^{16} a^{10}} - \frac{3b^6 c^3}{2^{11} a^9} + \frac{3b^5 c^2 d}{2^9 a^8} + \frac{b^4 c^4}{2^8 a^8} - \frac{3b^4 c^2 e}{2^7 a^7} - \frac{b^3 c^3 d}{2^5 a^7} + \frac{b^2 c^3 e}{2^3 a^6} + \frac{b^2 c^2 d^2}{2^4 a^6} - \frac{bc^2 de}{2a^5} + \frac{c^2 e^2}{a^4} \\
 &= \frac{3^4 b^{12}}{2^{22} a^{12}} - \frac{3^4 b^{10} c}{2^{18} a^{11}} + \frac{3^3 b^9 d}{2^{15} a^{10}} + \frac{3^2 13 b^8 c^2}{2^{16} a^{10}} - \frac{3^3 b^8 e}{2^{13} a^9} - \frac{3^2 b^7 cd}{2^{10} a^9} - \frac{3^2 b^6 c^3}{2^{11} a^9} + \frac{3^2 b^6 ce}{2^8 a^8} + \frac{3^2 b^6 d^2}{2^{10} a^8} \\
 &\quad + \frac{3 \cdot 5 b^5 c^2 d}{2^9 a^8} + \frac{b^4 c^4}{2^8 a^8} - \frac{3^2 b^5 de}{2^7 a^7} - \frac{3 \cdot 5 b^4 c^2 e}{2^7 a^7} - \frac{3b^4 cd^2}{2^6 a^7} - \frac{b^3 c^3 d}{2^5 a^7} + \frac{3^2 b^4 e^2}{2^6 a^6} + \frac{3b^3 cde}{2^3 a^6} + \frac{b^2 c^3 e}{2^3 a^6} \\
 &\quad + \frac{b^2 c^2 d^2}{2^4 a^6} - \frac{3b^2 ce^2}{2^2 a^5} - \frac{bc^2 de}{2a^5} + \frac{c^2 e^2}{a^4}
 \end{aligned}$$

$$\begin{aligned}
 A_1A_2^2A_3 &= \left(-\frac{3b^2}{2^3a^2} + \frac{c}{a}\right) \left(\frac{b^6}{2^6a^6} - \frac{b^4c}{2^3a^5} + \frac{b^3d}{2^2a^4} + \frac{b^2c^2}{2^2a^4} - \frac{bcd}{a^3} + \frac{d^2}{a^2}\right) \left(-\frac{3b^4}{2^8a^4} + \frac{b^2c}{2^4a^3} - \frac{bd}{2^2a^2} + \frac{e}{a}\right) \\
 &= \left(\frac{b^6}{2^6a^6} - \frac{b^4c}{2^3a^5} + \frac{b^3d}{2^2a^4} + \frac{b^2c^2}{2^2a^4} - \frac{bcd}{a^3} + \frac{d^2}{a^2}\right) \left(-\frac{3b^2}{2^3a^2} + \frac{c}{a}\right) \left(-\frac{3b^4}{2^8a^4} + \frac{b^2c}{2^4a^3} - \frac{bd}{2^2a^2} + \frac{e}{a}\right) \\
 &= \left(\frac{b^6}{2^6a^6} - \frac{b^4c}{2^3a^5} + \frac{b^3d}{2^2a^4} + \frac{b^2c^2}{2^2a^4} - \frac{bcd}{a^3} + \frac{d^2}{a^2}\right) \left(\frac{3^2b^6}{2^{11}a^6} - \frac{3b^4c}{2^7a^5} + \frac{3b^3d}{2^5a^4} - \frac{3b^2e}{2^3a^3} - \frac{3b^4c}{2^8a^5} + \frac{b^2c^2}{2^4a^4} - \frac{bcd}{2^2a^3} + \frac{ce}{a^2}\right) \\
 &= \frac{3^2b^{12}}{2^{17}a^{12}} - \frac{3b^{10}c}{2^{13}a^{11}} + \frac{3b^9d}{2^{11}a^{10}} - \frac{3b^8e}{2^9a^9} - \frac{3b^{10}c}{2^{14}a^{11}} + \frac{b^8c^2}{2^{10}a^{10}} - \frac{b^7cd}{2^8a^9} + \frac{b^6ce}{2^6a^8} \\
 &\quad - \frac{3^2b^{10}c}{2^{14}a^{11}} + \frac{3b^8c^2}{2^{10}a^{10}} - \frac{3b^7cd}{2^8a^9} + \frac{3b^6ce}{2^6a^8} + \frac{3b^8c^2}{2^{11}a^{10}} - \frac{b^6c^3}{2^7a^9} + \frac{b^5c^2d}{2^5a^8} - \frac{b^4c^2e}{2^3a^7} \\
 &\quad + \frac{3^2b^9d}{2^{13}a^{10}} - \frac{3b^7cd}{2^9a^9} + \frac{3b^6d^2}{2^7a^8} - \frac{3b^5de}{2^5a^7} - \frac{3b^7cd}{2^{10}a^9} + \frac{b^5c^2d}{2^6a^8} - \frac{b^4cd^2}{2^4a^7} + \frac{b^3cde}{2^2a^6} \\
 &\quad + \frac{3^2b^8c^2}{2^{13}a^{10}} - \frac{3b^6c^3}{2^9a^9} + \frac{3b^5c^2d}{2^7a^8} - \frac{3b^4c^2e}{2^5a^7} - \frac{3b^6c^3}{2^{10}a^9} + \frac{b^4c^4}{2^6a^8} - \frac{b^3c^3d}{2^4a^7} + \frac{b^2c^3e}{2^2a^6} \\
 &\quad - \frac{3^2b^7cd}{2^{11}a^9} + \frac{3b^5c^2d}{2^7a^8} - \frac{3b^4cd^2}{2^5a^7} + \frac{3b^3cde}{2^3a^6} + \frac{3b^5c^2d}{2^8a^8} - \frac{b^3c^3d}{2^4a^7} + \frac{b^2c^2d^2}{2^2a^6} - \frac{bc^2de}{a^5} \\
 &\quad + \frac{3^2b^6d^2}{2^{11}a^8} - \frac{3b^4cd^2}{2^7a^7} + \frac{3b^3d^3}{2^5a^6} - \frac{3b^2d^2e}{2^3a^5} - \frac{3b^4cd^2}{2^8a^7} + \frac{b^2c^2d^2}{2^4a^6} - \frac{bcd^3}{2^2a^5} + \frac{cd^2e}{a^4} \\
 &= \frac{3^2b^{12}}{2^{17}a^{12}} - \frac{3^2b^{10}c}{2^{13}a^{11}} + \frac{3 \cdot 7b^9d}{2^{13}a^{10}} + \frac{53b^8c^2}{2^{13}a^{10}} - \frac{3b^8e}{2^9a^9} - \frac{59b^7cd}{2^{11}a^9} - \frac{17b^6c^3}{2^{10}a^9} + \frac{b^6ce}{2^4a^8} + \frac{3 \cdot 19b^6d^2}{2^{11}a^8} \\
 &\quad + \frac{3^3b^5c^2d}{2^8a^8} + \frac{b^4c^4}{2^6a^8} - \frac{3b^5de}{2^5a^7} - \frac{7b^4c^2e}{2^5a^7} - \frac{7^2b^4cd^2}{2^8a^7} - \frac{b^3c^3d}{2^3a^7} + \frac{5b^3cde}{2^3a^6} + \frac{3b^3d^3}{2^5a^6} + \frac{b^2c^3e}{2^2a^6} \\
 &\quad + \frac{5b^2c^2d^2}{2^4a^6} - \frac{3b^2d^2e}{2^3a^5} - \frac{bc^2de}{a^5} - \frac{bcd^3}{2^2a^5} + \frac{cd^2e}{a^4}
 \end{aligned}$$

式 (12) より、

$$\begin{aligned}
U &= 2^3 A_1^3 - 2^5 3^2 A_1 A_3 + 2^2 3^3 A_2^2 \\
&= 2^3 \left(-\frac{3^3 b^6}{2^9 a^6} + \frac{3^3 b^4 c}{2^6 a^5} - \frac{3^2 b^2 c^2}{2^3 a^4} + \frac{c^3}{a^3} \right) \\
&\quad - 2^5 3^2 \left(\frac{3^2 b^6}{2^{11} a^6} - \frac{3^2 b^4 c}{2^8 a^5} + \frac{3 b^3 d}{2^5 a^4} + \frac{b^2 c^2}{2^4 a^4} - \frac{3 b^2 e}{2^3 a^3} - \frac{bcd}{2^2 a^3} + \frac{ce}{a^2} \right) \\
&\quad + 2^2 3^3 \left(\frac{b^6}{2^6 a^6} - \frac{b^4 c}{2^3 a^5} + \frac{b^3 d}{2^2 a^4} + \frac{b^2 c^2}{2^2 a^4} - \frac{bcd}{a^3} + \frac{d^2}{a^2} \right) \\
&= -\frac{3^3 b^6}{2^6 a^6} + \frac{3^3 b^4 c}{2^3 a^5} - \frac{3^2 b^2 c^2}{a^4} + \frac{2^3 c^3}{a^3} \\
&\quad - \frac{3^4 b^6}{2^6 a^6} + \frac{3^4 b^4 c}{2^3 a^5} - \frac{3^3 b^3 d}{a^4} - \frac{2 \cdot 3^2 b^2 c^2}{a^4} + \frac{2^2 3^3 b^2 e}{a^3} + \frac{2^3 3^2 bcd}{a^3} - \frac{2^5 3^2 ce}{a^2} \\
&\quad + \frac{3^3 b^6}{2^4 a^6} - \frac{3^3 b^4 c}{2 a^5} + \frac{3^3 b^3 d}{a^4} + \frac{3^3 b^2 c^2}{a^4} - \frac{2^2 3^3 bcd}{a^3} + \frac{2^2 3^3 d^2}{a^2} \\
&= \frac{2^2 3^3 b^2 e}{a^3} - \frac{2^2 3^2 bcd}{a^3} + \frac{2^3 c^3}{a^3} - \frac{2^5 3^2 ce}{a^2} + \frac{2^2 3^3 d^2}{a^2} \\
&\quad \because (A) = (B) = (C1) = (C2) = 0 \\
&= \frac{2^2}{a^3} (3^3 b^2 e - 3^2 bcd + 2c^3 - 2^3 3^2 ace + 3^3 ad^2) \\
&\quad \text{並べ直して} \\
&= \frac{2^2}{a^3} (-2^3 3^2 ace + 3^3 ad^2 + 3^3 b^2 e - 3^2 bcd + 2c^3) \\
\therefore U &= \frac{4}{a^3} (-72ace + 27ad^2 + 27b^2e - 9bcd + 2c^3) \tag{18}
\end{aligned}$$

式 (13) より、

$$\begin{aligned}
 V &= -2^4 3 A_1^4 A_3 + 2^2 3 A_1^3 A_2^2 + 2^7 3 A_1^2 A_3^2 - 2^4 3^3 A_1 A_2^2 A_3 + 3^4 A_2^4 - 2^8 3 A_3^3 \\
 &= -2^4 3 \left(-\frac{3^5 b^{12}}{2^{20} a^{12}} + \frac{3^5 b^{10} c}{2^{16} a^{11}} - \frac{3^4 b^9 d}{2^{14} a^{10}} - \frac{3^3 7 b^8 c^2}{2^{13} a^{10}} + \frac{3^4 b^8 e}{2^{12} a^9} + \frac{3^3 b^7 c d}{2^9 a^9} + \frac{3^2 b^6 c^3}{2^7 a^9} - \frac{3^3 b^6 c e}{2^7 a^8} \right. \\
 &\quad \left. - \frac{3^3 b^5 c^2 d}{2^7 a^8} - \frac{3^3 b^4 c^4}{2^8 a^8} + \frac{3^3 b^4 c^2 e}{2^5 a^7} + \frac{3 b^3 c^3 d}{2^3 a^7} + \frac{b^2 c^5}{2^4 a^7} - \frac{3 b^2 c^3 e}{2 a^6} - \frac{b c^4 d}{2^2 a^6} + \frac{c^4 e}{a^5} \right) \\
 &+ 2^2 3 \left(-\frac{3^3 b^{12}}{2^{15} a^{12}} + \frac{3^3 b^{10} c}{2^{11} a^{11}} - \frac{3^3 b^9 d}{2^{11} a^{10}} - \frac{3^2 19 b^8 c^2}{2^{11} a^{10}} + \frac{3^4 b^7 c d}{2^9 a^9} + \frac{67 b^6 c^3}{2^8 a^9} - \frac{3^3 b^6 d^2}{2^9 a^8} - \frac{3^2 5 b^5 c^2 d}{2^6 a^8} \right. \\
 &\quad \left. - \frac{13 b^4 c^4}{2^5 a^8} + \frac{3^3 b^4 c d^2}{2^6 a^7} + \frac{11 b^3 c^3 d}{2^3 a^7} + \frac{b^2 c^5}{2^2 a^7} - \frac{3^2 b^2 c^2 d^2}{2^3 a^6} - \frac{b c^4 d}{a^6} + \frac{c^3 d^2}{a^5} \right) \\
 &+ 2^7 3 \left(\frac{3^4 b^{12}}{2^{22} a^{12}} - \frac{3^4 b^{10} c}{2^{18} a^{11}} + \frac{3^3 b^9 d}{2^{15} a^{10}} + \frac{3^2 13 b^8 c^2}{2^{16} a^{10}} - \frac{3^3 b^8 e}{2^{13} a^9} - \frac{3^2 b^7 c d}{2^{10} a^9} - \frac{3^2 b^6 c^3}{2^{11} a^9} + \frac{3^2 b^6 c e}{2^8 a^8} + \frac{3^2 b^6 d^2}{2^{10} a^8} \right. \\
 &\quad + \frac{3 \cdot 5 b^5 c^2 d}{2^9 a^8} + \frac{b^4 c^4}{2^8 a^8} - \frac{3^2 b^5 d e}{2^7 a^7} - \frac{3 \cdot 5 b^4 c^2 e}{2^7 a^7} - \frac{3 b^4 c d^2}{2^6 a^7} - \frac{b^3 c^3 d}{2^5 a^7} + \frac{3^2 b^4 e^2}{2^6 a^6} + \frac{3 b^3 c d e}{2^3 a^6} + \frac{b^2 c^3 e}{2^3 a^6} \\
 &\quad \left. + \frac{b^2 c^2 d^2}{2^4 a^6} - \frac{3 b^2 c e^2}{2^2 a^5} - \frac{b c^2 d e}{2 a^5} + \frac{c^2 e^2}{a^4} \right) \\
 &- 2^4 3^3 \left(\frac{3^2 b^{12}}{2^{17} a^{12}} - \frac{3^2 b^{10} c}{2^{13} a^{11}} + \frac{3 \cdot 7 b^9 d}{2^{13} a^{10}} + \frac{53 b^8 c^2}{2^{13} a^{10}} - \frac{3 b^8 e}{2^9 a^9} - \frac{59 b^7 c d}{2^{11} a^9} - \frac{17 b^6 c^3}{2^{10} a^9} + \frac{b^6 c e}{2^4 a^8} + \frac{3 \cdot 19 b^6 d^2}{2^{11} a^8} \right. \\
 &\quad + \frac{3^3 b^5 c^2 d}{2^8 a^8} + \frac{b^4 c^4}{2^6 a^8} - \frac{3 b^5 d e}{2^5 a^7} - \frac{7 b^4 c^2 e}{2^5 a^7} - \frac{7^2 b^4 c d^2}{2^8 a^7} - \frac{b^3 c^3 d}{2^3 a^7} + \frac{5 b^3 c d e}{2^3 a^6} + \frac{3 b^3 d^3}{2^5 a^6} + \frac{b^2 c^3 e}{2^2 a^6} \\
 &\quad \left. + \frac{5 b^2 c^2 d^2}{2^4 a^6} - \frac{3 b^2 d^2 e}{2^3 a^5} - \frac{b c^2 d e}{a^5} - \frac{b c d^3}{2^2 a^5} + \frac{c d^2 e}{a^4} \right) \\
 &+ 3^4 \left(\frac{b^{12}}{2^{12} a^{12}} - \frac{b^{10} c}{2^8 a^{11}} + \frac{b^9 d}{2^7 a^{10}} + \frac{3 b^8 c^2}{2^7 a^{10}} - \frac{3 b^7 c d}{2^5 a^9} - \frac{b^6 c^3}{2^4 a^9} + \frac{3 b^6 d^2}{2^5 a^8} + \frac{3 b^5 c^2 d}{2^3 a^8} + \frac{b^4 c^4}{2^4 a^8} \right. \\
 &\quad \left. - \frac{3 b^4 c d^2}{2^2 a^7} - \frac{b^3 c^3 d}{2 a^7} + \frac{b^3 d^3}{2 a^6} + \frac{3 b^2 c^2 d^2}{2 a^6} - \frac{2 b c d^3}{a^5} + \frac{d^4}{a^4} \right) \\
 &- 2^8 3 \left(-\frac{3^3 b^{12}}{2^{24} a^{12}} + \frac{3^3 b^{10} c}{2^{20} a^{11}} - \frac{3^3 b^9 d}{2^{18} a^{10}} - \frac{3^2 b^8 c^2}{2^{16} a^{10}} + \frac{3^3 b^8 e}{2^{16} a^9} + \frac{3^2 b^7 c d}{2^{13} a^9} + \frac{b^6 c^3}{2^{12} a^9} - \frac{3^2 b^6 c e}{2^{11} a^8} - \frac{3^2 b^6 d^2}{2^{12} a^8} - \frac{3 b^5 c^2 d}{2^{10} a^8} \right. \\
 &\quad \left. + \frac{3^2 b^5 d e}{2^9 a^7} + \frac{3 b^4 c^2 e}{2^8 a^7} + \frac{3 b^4 c d^2}{2^8 a^7} - \frac{3^2 b^4 e^2}{2^8 a^6} - \frac{3 b^3 c d e}{2^5 a^6} - \frac{b^3 d^3}{2^6 a^6} + \frac{3 b^2 c e^2}{2^4 a^5} + \frac{3 b^2 d^2 e}{2^4 a^5} - \frac{3 b d e^2}{2^2 a^4} + \frac{e^3}{a^3} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{3^6 b^{12}}{2^{16} a^{12}} - \frac{3^6 b^{10} c}{2^{12} a^{11}} + \frac{3^5 b^9 d}{2^{10} a^{10}} + \frac{3^4 7 b^8 c^2}{2^9 a^{10}} - \frac{3^5 b^8 e}{2^8 a^9} - \frac{3^4 b^7 c d}{2^5 a^9} - \frac{3^3 b^6 c^3}{2^3 a^9} + \frac{3^4 b^6 c e}{2^3 a^8} \\
 &+ \frac{3^4 b^5 c^2 d}{2^3 a^8} + \frac{3^4 b^4 c^4}{2^4 a^8} - \frac{3^4 b^4 c^2 e}{2 a^7} - \frac{2 \cdot 3^2 b^3 c^3 d}{a^7} - \frac{3 b^2 c^5}{a^7} + \frac{2^3 3^2 b^2 c^3 e}{a^6} + \frac{2^2 3 b c^4 d}{a^6} - \frac{2^4 3 c^4 e}{a^5} \\
 &- \frac{3^4 b^{12}}{2^{13} a^{12}} + \frac{3^4 b^{10} c}{2^9 a^{11}} - \frac{3^4 b^9 d}{2^9 a^{10}} - \frac{3^3 19 b^8 c^2}{2^9 a^{10}} + \frac{3^5 b^7 c d}{2^7 a^9} + \frac{3 \cdot 67 b^6 c^3}{2^6 a^9} - \frac{3^4 b^6 d^2}{2^7 a^8} - \frac{3^3 5 b^5 c^2 d}{2^4 a^8} \\
 &- \frac{3 \cdot 13 b^4 c^4}{2^3 a^8} + \frac{3^4 b^4 c d^2}{2^4 a^7} + \frac{3 \cdot 11 b^3 c^3 d}{2 a^7} + \frac{3 b^2 c^5}{a^7} - \frac{3^3 b^2 c^2 d^2}{2 a^6} - \frac{2^2 3 b c^4 d}{a^6} + \frac{2^2 3 c^3 d^2}{a^5} \\
 &+ \frac{3^5 b^{12}}{2^{15} a^{12}} - \frac{3^5 b^{10} c}{2^{11} a^{11}} + \frac{3^4 b^9 d}{2^8 a^{10}} + \frac{3^3 13 b^8 c^2}{2^9 a^{10}} - \frac{3^4 b^8 e}{2^6 a^9} - \frac{3^3 b^7 c d}{2^3 a^9} - \frac{3^3 b^6 c^3}{2^4 a^9} + \frac{3^3 b^6 c e}{2 a^8} + \frac{3^3 b^6 d^2}{2^3 a^8} \\
 &+ \frac{3^2 5 b^5 c^2 d}{2^2 a^8} + \frac{3 b^4 c^4}{2 a^8} - \frac{3^3 b^5 d e}{a^7} - \frac{3^2 5 b^4 c^2 e}{a^7} - \frac{2 \cdot 3^2 b^4 c d^2}{a^7} - \frac{2^2 3 b^3 c^3 d}{a^7} + \frac{2 \cdot 3^3 b^4 e^2}{a^6} + \frac{2^4 3^2 b^3 c d e}{a^6} + \frac{2^4 3 b^2 c^3 e}{a^6} \\
 &+ \frac{2^3 3 b^2 c^2 d^2}{a^6} - \frac{2^5 3^2 b^2 c e^2}{a^5} - \frac{2^6 3 b c^2 d e}{a^5} + \frac{2^7 3 c^2 e^2}{a^4} \\
 &- \frac{3^5 b^{12}}{2^{13} a^{12}} + \frac{3^5 b^{10} c}{2^9 a^{11}} - \frac{3^4 7 b^9 d}{2^9 a^{10}} - \frac{3^3 53 b^8 c^2}{2^9 a^{10}} + \frac{3^4 b^8 e}{2^5 a^9} + \frac{3^3 59 b^7 c d}{2^7 a^9} + \frac{3^3 17 b^6 c^3}{2^6 a^9} - \frac{3^3 b^6 c e}{a^8} - \frac{3^4 19 b^6 d^2}{2^7 a^8} \\
 &- \frac{3^6 b^5 c^2 d}{2^4 a^8} - \frac{3^3 b^4 c^4}{2^2 a^8} + \frac{3^4 b^5 d e}{2 a^7} + \frac{3^3 7 b^4 c^2 e}{2 a^7} + \frac{3^3 7^2 b^4 c d^2}{2^4 a^7} + \frac{2 \cdot 3^3 b^3 c^3 d}{a^7} - \frac{2 \cdot 3^3 5 b^3 c d e}{a^6} - \frac{3^4 b^3 d^3}{2 a^6} - \frac{2^2 3^3 b^2 c^3 e}{a^6} \\
 &- \frac{3^3 5 b^2 c^2 d^2}{a^6} + \frac{2 \cdot 3^4 b^2 d^2 e}{a^5} + \frac{2^4 3^3 b c^2 d e}{a^5} + \frac{2^2 3^3 b c d^3}{a^5} - \frac{2^4 3^3 c d^2 e}{a^4} \\
 &+ \frac{3^4 b^{12}}{2^{12} a^{12}} - \frac{3^4 b^{10} c}{2^8 a^{11}} + \frac{3^4 b^9 d}{2^7 a^{10}} + \frac{3^5 b^8 c^2}{2^7 a^{10}} - \frac{3^5 b^7 c d}{2^5 a^9} - \frac{3^4 b^6 c^3}{2^4 a^9} + \frac{3^5 b^6 d^2}{2^5 a^8} + \frac{3^5 b^5 c^2 d}{2^3 a^8} + \frac{3^4 b^4 c^4}{2^4 a^8} \\
 &- \frac{3^5 b^4 c d^2}{2^2 a^7} - \frac{3^4 b^3 c^3 d}{2 a^7} + \frac{3^4 b^3 d^3}{2 a^6} + \frac{3^5 b^2 c^2 d^2}{2 a^6} - \frac{2 \cdot 3^4 b c d^3}{a^5} + \frac{3^4 d^4}{a^4} \\
 &+ \frac{3^4 b^{12}}{2^{16} a^{12}} - \frac{3^4 b^{10} c}{2^{12} a^{11}} + \frac{3^4 b^9 d}{2^{10} a^{10}} + \frac{3^3 b^8 c^2}{2^8 a^{10}} - \frac{3^4 b^8 e}{2^8 a^9} - \frac{3^3 b^7 c d}{2^5 a^9} - \frac{3 b^6 c^3}{2^4 a^9} + \frac{3^3 b^6 c e}{2^3 a^8} + \frac{3^3 b^6 d^2}{2^4 a^8} + \frac{3^2 b^5 c^2 d}{2^2 a^8} \\
 &- \frac{3^3 b^5 d e}{2 a^7} - \frac{3^2 b^4 c^2 e}{a^7} - \frac{3^2 b^4 c d^2}{a^7} + \frac{3^3 b^4 e^2}{a^6} + \frac{2^3 3^2 b^3 c d e}{a^6} + \frac{2^2 3 b^3 d^3}{a^6} - \frac{2^4 3^2 b^2 c e^2}{a^5} - \frac{2^4 3^2 b^2 d^2 e}{a^5} + \frac{2^6 3^2 b d e^2}{a^4} - \frac{2^8 3 e^3}{a^3} \\
 &= \frac{3^4 b^4 e^2}{a^6} - \frac{2 \cdot 3^3 b^3 c d e}{a^6} + \frac{2^2 3 b^3 d^3}{a^6} + \frac{2^2 3 b^2 c^3 e}{a^6} - \frac{3 b^2 c^2 d^2}{a^6} - \frac{2^4 3^3 b^2 c e^2}{a^5} + \frac{2 \cdot 3^2 b^2 d^2 e}{a^5} + \frac{2^4 3 \cdot 5 b c^2 d e}{a^5} \\
 &- \frac{2 \cdot 3^3 b c d^3}{a^5} - \frac{2^4 3 c^4 e}{a^5} + \frac{2^2 3 c^3 d^2}{a^5} + \frac{2^6 3^2 b d e^2}{a^4} + \frac{2^7 3 c^2 e^2}{a^4} \\
 &- \frac{2^4 3^3 c d^2 e}{a^4} + \frac{3^4 d^4}{a^4} - \frac{2^8 3 e^3}{a^3}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{3}{a^6} (3^3 b^4 e^2 - 2 \cdot 3^2 b^3 c d e + 2^2 b^3 d^3 + 2^2 b^2 c^3 e - b^2 c^2 d^2 - 2^4 3^2 a b^2 c e^2 + 2 \cdot 3 a b^2 d^2 e + 2^4 5 a b c^2 d e \\
&\quad - 2 \cdot 3^2 a b c d^3 - 2^4 a c^4 e + 2^2 a c^3 d^2 + 2^6 3 a^2 b d e^2 + 2^7 a^2 c^2 e^2 - 2^4 3^2 a^2 c d^2 e + 3^3 a^2 d^4 - 2^8 a^3 e^3) \\
&\text{並べ直して} \\
&= \frac{3}{a^6} (-2^8 a^3 e^3 + 2^6 3 a^2 b d e^2 + 2^7 a^2 c^2 e^2 - 2^4 3^2 a^2 c d^2 e + 3^3 a^2 d^4 - 2^4 3^2 a b^2 c e^2 + 2 \cdot 3 a b^2 d^2 e + 2^4 5 a b c^2 d e \\
&\quad - 2 \cdot 3^2 a b c d^3 - 2^4 a c^4 e + 2^2 a c^3 d^2 + 3^3 b^4 e^2 - 2 \cdot 3^2 b^3 c d e + 2^2 b^3 d^3 + 2^2 b^2 c^3 e - b^2 c^2 d^2) \\
\therefore V &= \frac{3}{a^6} (-256 a^3 e^3 + 192 a^2 b d e^2 + 128 a^2 c^2 e^2 - 144 a^2 c d^2 e + 27 a^2 d^4 - 144 a b^2 c e^2 + 6 a b^2 d^2 e + 80 a b c^2 d e \\
&\quad - 18 a b c d^3 - 16 a c^4 e + 4 a c^3 d^2 + 27 b^4 e^2 - 18 b^3 c d e + 4 b^3 d^3 + 4 b^2 c^3 e - b^2 c^2 d^2) \quad (20)
\end{aligned}$$

三次方程式由来の条件式 (17) は、

$$\begin{aligned}
S^{\frac{1}{3}} T^{\frac{1}{3}} &= 2^2 A_1^2 + 2^4 3 A_3 \\
&= 2^2 \left(\frac{3^2 b^4}{2^6 a^4} - \frac{3 b^2 c}{2^2 a^3} + \frac{c^2}{a^2} \right) + 2^4 3 \left(-\frac{3 b^4}{2^8 a^4} + \frac{b^2 c}{2^4 a^3} - \frac{b d}{2^2 a^2} + \frac{e}{a} \right) \\
&= \frac{3^2 b^4}{2^4 a^4} - \frac{3 b^2 c}{a^3} + \frac{2^2 c^2}{a^2} - \frac{3^2 b^4}{2^4 a^4} + \frac{3 b^2 c}{a^3} - \frac{2^2 3 b d}{a^2} + \frac{2^4 3 e}{a} \\
&= \frac{2^2 c^2}{a^2} - \frac{2^2 3 b d}{a^2} + \frac{2^4 3 e}{a} \\
&= \frac{4}{a^2} (12 a e - 3 b d + c^2)
\end{aligned}$$

よって、

$$X = a^3 U \quad Y = a^6 V \quad P = a^3 S \quad Q = a^3 T$$

と置き直して、

$$\begin{aligned}
S &= U + 12\sqrt{V} = \frac{X}{a^3} + 12\sqrt{\frac{Y}{a^6}} = \frac{1}{a^3} (X + 12\sqrt{Y}) \\
T &= U - 12\sqrt{V} = \frac{X}{a^3} - 12\sqrt{\frac{Y}{a^6}} = \frac{1}{a^3} (X - 12\sqrt{Y}) \\
P &= X + 12\sqrt{Y} \\
Q &= X - 12\sqrt{Y}
\end{aligned}$$

$S^{\frac{1}{3}} T^{\frac{1}{3}}$ を $P \cdot Q$ で書き直すと、

$$P^{\frac{1}{3}} Q^{\frac{1}{3}} = 4 (12 a e - 3 b d + c^2)$$

$$\beta_i = \frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{s_i}{6} S^{\frac{1}{3}} + \frac{t_i}{6} T^{\frac{1}{3}} = \frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{s_i}{6a} P^{\frac{1}{3}} + \frac{t_i}{6a} Q^{\frac{1}{3}}$$

但し、

$$\begin{aligned}
1) \quad (s_i, t_i) &= (1, 1), (\omega, \omega^2), (\omega^2, \omega) & P^{\frac{1}{3}} Q^{\frac{1}{3}} &= 4 (12 a e - 3 b d + c^2) \\
2) \quad (s_i, t_i) &= (1, \omega^2), (\omega, \omega), (\omega^2, 1) & P^{\frac{1}{3}} Q^{\frac{1}{3}} &= 4 \omega (12 a e - 3 b d + c^2) \\
3) \quad (s_i, t_i) &= (1, \omega), (\omega, 1), (\omega^2, \omega^2) & P^{\frac{1}{3}} Q^{\frac{1}{3}} &= 4 \omega^2 (12 a e - 3 b d + c^2)
\end{aligned}$$

$$R = -24ac + 9b^2$$

と置くと、

$$\begin{aligned} x_{ik} &= -\frac{b}{4a} + \frac{1}{2} \left(l_k \sqrt{\beta_1} + m_k \sqrt{\beta_2} + n_k \sqrt{\beta_3} \right) \\ &= -\frac{b}{4a} + \frac{1}{2} \left(l_k \sqrt{\frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{s_i}{6a} P^{\frac{1}{3}} + \frac{t_i}{6a} Q^{\frac{1}{3}}} + m_k \sqrt{\frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{s_i}{6a} P^{\frac{1}{3}} + \frac{t_i}{6a} Q^{\frac{1}{3}}} + n_k \sqrt{\frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{s_i}{6a} P^{\frac{1}{3}} + \frac{t_i}{6a} Q^{\frac{1}{3}}} \right) \\ &= -\frac{b}{4a} + \frac{1}{12a} \left(l_k \sqrt{9b^2 - 24ac + 6as_i P^{\frac{1}{3}} + 6at_i Q^{\frac{1}{3}}} \right. \\ &\quad \left. + m_k \sqrt{9b^2 - 24ac + 6as_i P^{\frac{1}{3}} + 6at_i Q^{\frac{1}{3}}} + n_k \sqrt{9b^2 - 24ac + 6as_i P^{\frac{1}{3}} + 6at_i Q^{\frac{1}{3}}} \right) \\ &= -\frac{b}{4a} + \frac{1}{12a} \left(l_k \sqrt{R + 6as_i P^{\frac{1}{3}} + 6at_i Q^{\frac{1}{3}}} + m_k \sqrt{R + 6as_i P^{\frac{1}{3}} + 6at_i Q^{\frac{1}{3}}} + n_k \sqrt{R + 6as_i P^{\frac{1}{3}} + 6at_i Q^{\frac{1}{3}}} \right) \end{aligned}$$

$\sqrt{\beta_1} \sqrt{\beta_2} \sqrt{\beta_3}$ に関する条件式は、

$(l_k, m_k, n_k) = (1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)$ の場合、

$$\begin{aligned} \sqrt{\beta_1} \sqrt{\beta_2} \sqrt{\beta_3} &= -\frac{b^3}{2^3 a^3} + \frac{bc}{2a^2} - \frac{d}{a} \\ \sqrt{\frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{s_i}{6a} P^{\frac{1}{3}} + \frac{t_i}{6a} Q^{\frac{1}{3}}} \cdot \sqrt{\frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{s_i}{6a} P^{\frac{1}{3}} + \frac{t_i}{6a} Q^{\frac{1}{3}}} \cdot \sqrt{\frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{s_i}{6a} P^{\frac{1}{3}} + \frac{t_i}{6a} Q^{\frac{1}{3}}} \\ &= -\frac{b^3}{2^3 a^3} + \frac{bc}{2a^2} - \frac{d}{a} \\ \sqrt{3^2 b^2 - 2^3 3ac + 2 \cdot 3as_i P^{\frac{1}{3}} + 2 \cdot 3at_i Q^{\frac{1}{3}}} \cdot \sqrt{3^2 b^2 - 2^3 3ac + 2 \cdot 3as_i P^{\frac{1}{3}} + 2 \cdot 3at_i Q^{\frac{1}{3}}} \\ &\quad \times \sqrt{3^2 b^2 - 2^3 3ac + 2 \cdot 3as_i P^{\frac{1}{3}} + 2 \cdot 3at_i Q^{\frac{1}{3}}} = -3^3 b^3 + 2^2 3^3 abc - 2^3 3^3 a^2 d \end{aligned}$$

よって、

$$\sqrt{R + 6as_i P^{\frac{1}{3}} + 6at_i Q^{\frac{1}{3}}} \cdot \sqrt{R + 6as_i P^{\frac{1}{3}} + 6at_i Q^{\frac{1}{3}}} \cdot \sqrt{R + 6as_i P^{\frac{1}{3}} + 6at_i Q^{\frac{1}{3}}} = 27(-8a^2 d + 4abc - b^3)$$

かつ、

$$\begin{aligned} (s_i, t_i) &= (1, 1), (\omega, \omega^2), (\omega^2, \omega) \text{ の場合、} & P^{\frac{1}{3}} Q^{\frac{1}{3}} &= 4(12ae - 3bd + c^2) \\ (s_i, t_i) &= (1, \omega^2), (\omega, \omega), (\omega^2, 1) \text{ の場合、} & P^{\frac{1}{3}} Q^{\frac{1}{3}} &= 4\omega(12ae - 3bd + c^2) \\ (s_i, t_i) &= (1, \omega), (\omega, 1), (\omega^2, \omega^2) \text{ の場合、} & P^{\frac{1}{3}} Q^{\frac{1}{3}} &= 4\omega^2(12ae - 3bd + c^2) \end{aligned}$$

$(l_k, m_k, n_k) = (1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)$ の場合、

$$\sqrt{\beta_1} \sqrt{\beta_2} \sqrt{\beta_3} = \frac{b^3}{2^3 a^3} - \frac{bc}{2a^2} + \frac{d}{a}$$

同様に、

$$\sqrt{R + 6as_i P^{\frac{1}{3}} + 6at_i Q^{\frac{1}{3}}} \cdot \sqrt{R + 6as_i P^{\frac{1}{3}} + 6at_i Q^{\frac{1}{3}}} \cdot \sqrt{R + 6as_i P^{\frac{1}{3}} + 6at_i Q^{\frac{1}{3}}} = 27(8a^2 d - 4abc + b^3)$$

かつ、

$$\begin{aligned} (s_i, t_i) = (1, 1), (\omega, \omega^2), (\omega^2, \omega) \text{ の場合、} & P^{\frac{1}{3}}Q^{\frac{1}{3}} = 4(12ae - 3bd + c^2) \\ (s_i, t_i) = (1, \omega^2), (\omega, \omega), (\omega^2, 1) \text{ の場合、} & P^{\frac{1}{3}}Q^{\frac{1}{3}} = 4\omega(12ae - 3bd + c^2) \\ (s_i, t_i) = (1, \omega), (\omega, 1), (\omega^2, \omega^2) \text{ の場合、} & P^{\frac{1}{3}}Q^{\frac{1}{3}} = 4\omega^2(12ae - 3bd + c^2) \end{aligned}$$

以上をまとめると、

4次方程式 $ax^4 + bx^3 + cx^2 + dx + e = 0$ (a, b, c, d, e は実数定数で $a \neq 0$) の解は、

$$R = -24ac + 9b^2$$

$$X = 4(-72ace + 27ad^2 + 27b^2e - 9bcd + 2c^3)$$

$$Y = 3(-256a^3e^3 + 192a^2bde^2 + 128a^2c^2e^2 - 144a^2cd^2e + 27a^2d^4 - 144ab^2ce^2 + 6ab^2d^2e + 80abc^2de - 18abcd^3 - 16ac^4e + 4ac^3d^2 + 27b^4e^2 - 18b^3cde + 4b^3d^3 + 4b^2c^3e - b^2c^2d^2)$$

$$P = X + 12\sqrt{Y} \qquad \omega = \frac{-1 + \sqrt{3}i}{2}$$

$$Q = X - 12\sqrt{Y} \qquad \omega^2 = \frac{-1 - \sqrt{3}i}{2}$$

$$L_i = \sqrt{R + 6as_iP^{\frac{1}{3}} + 6at_iQ^{\frac{1}{3}}} \qquad M_i = \sqrt{R + 6as_iP^{\frac{1}{3}} + 6at_iQ^{\frac{1}{3}}} \qquad N_i = \sqrt{R + 6as_iP^{\frac{1}{3}} + 6at_iQ^{\frac{1}{3}}}$$

と置いて、

$$x_{ik} = -\frac{b}{4a} + \frac{1}{12a}(l_kL_i + m_kM_i + n_kN_i)$$

但し ($k = 1$)、

$$\begin{array}{cccc} & x_1 & x_2 & x_3 & x_4 \\ (l_1, m_1, n_1) = & (1, 1, 1), & (1, -1, -1), & (-1, 1, -1), & (-1, -1, 1) \\ L_iM_iN_i = & 27(-8a^2d + 4abc - b^3) & & & \end{array}$$

$$\begin{array}{cccc} & L_i & M_i & N_i & \\ \text{かつ } (i = 1) : & (s_1, t_1) = & (1, 1), & (\omega, \omega^2), & (\omega^2, \omega) \Rightarrow P^{\frac{1}{3}}Q^{\frac{1}{3}} = 4(12ae - 3bd + c^2) \\ \text{又は } (i = 2) : & (s_2, t_2) = & (1, \omega^2), & (\omega, \omega), & (\omega^2, 1) \Rightarrow P^{\frac{1}{3}}Q^{\frac{1}{3}} = 4\omega(12ae - 3bd + c^2) \\ \text{又は } (i = 3) : & (s_3, t_3) = & (1, \omega), & (\omega, 1), & (\omega^2, \omega^2) \Rightarrow P^{\frac{1}{3}}Q^{\frac{1}{3}} = 4\omega^2(12ae - 3bd + c^2) \end{array}$$

又は ($k = 2$)、

$$\begin{array}{cccc} & x_1 & x_2 & x_3 & x_4 \\ (l_2, m_2, n_2) = & (1, 1, -1), & (1, -1, 1), & (-1, 1, 1), & (-1, -1, -1) \\ L_iM_iN_i = & 27(8a^2d - 4abc + b^3) & & & \end{array}$$

$$\begin{array}{cccc} & L_i & M_i & N_i & \\ \text{かつ } (i = 1) : & (s_1, t_1) = & (1, 1), & (\omega, \omega^2), & (\omega^2, \omega) \Rightarrow P^{\frac{1}{3}}Q^{\frac{1}{3}} = -4(12ae - 3bd + c^2) \\ \text{又は } (i = 2) : & (s_2, t_2) = & (1, \omega^2), & (\omega, \omega), & (\omega^2, 1) \Rightarrow P^{\frac{1}{3}}Q^{\frac{1}{3}} = -4\omega(12ae - 3bd + c^2) \\ \text{又は } (i = 3) : & (s_3, t_3) = & (1, \omega), & (\omega, 1), & (\omega^2, \omega^2) \Rightarrow P^{\frac{1}{3}}Q^{\frac{1}{3}} = -4\omega^2(12ae - 3bd + c^2) \end{array}$$

合計 6 通りの組合せがあるが、どれも数学的には同じで、四次関数の答えとして 1 セット 4 つの (l, m, n) を採用すれば良い (例えば $k = 1, i = 1$)。

V の計算式 (19) の係数部計算詳細 (分子は最大公約数、分母は最小公倍数で括りだす)
 項目数が多い場合はプラスの項目 (⊕) とマイナスの項目 (⊖) で分けて計算すると楽。

(A)	$\frac{3^6}{2^{16}} - \frac{3^4}{2^{13}} + \frac{3^5}{2^{15}} - \frac{3^5}{2^{13}} + \frac{3^4}{2^{12}} + \frac{3^4}{2^{16}}$	$\left\{ \begin{array}{l} \oplus \frac{3^6}{2^{16}} + \frac{3^5}{2^{15}} + \frac{3^4}{2^{12}} + \frac{3^4}{2^{16}} = \frac{3^4}{2^{16}} \overbrace{(9+6+16+1)}^{32=2^5} = \frac{3^4}{2^{11}} \\ \ominus \frac{3^4}{2^{13}} + \frac{3^5}{2^{13}} = \frac{3^4}{2^{13}} (1+3) = \frac{3^4}{2^{11}} \end{array} \right\} = 0$
(B)	$-\frac{3^6}{2^{12}} + \frac{3^4}{2^9} - \frac{3^5}{2^{11}} + \frac{3^5}{2^9} - \frac{3^4}{2^8} - \frac{3^4}{2^{12}}$	$\left\{ \begin{array}{l} \oplus \frac{3^4}{2^9} + \frac{3^5}{2^9} = \frac{3^4}{2^9} (1+3) = \frac{3^4}{2^7} \\ \ominus \frac{3^6}{2^{12}} + \frac{3^5}{2^{11}} + \frac{3^4}{2^8} + \frac{3^4}{2^{12}} = \frac{3^4}{2^{12}} (9+6+16+1) = \frac{3^4}{2^7} \end{array} \right\} = 0$
(C1)	$\frac{3^5}{2^{10}} - \frac{3^4}{2^9} + \frac{3^4}{a^8} - \frac{3^{47}}{2^9} + \frac{3^4}{2^7} + \frac{3^4}{2^{10}}$	$\left\{ \begin{array}{l} \oplus \frac{3^5}{2^{10}} + \frac{3^4}{2^8} + \frac{3^4}{2^7} + \frac{3^4}{2^{10}} = \frac{3^4}{2^{10}} (3+4+8+1) = \frac{3^4}{2^6} \\ \ominus \frac{3^4}{2^9} + \frac{3^{47}}{2^9} = \frac{3^4}{2^9} (1+7) = \frac{3^4}{2^6} \end{array} \right\} = 0$
(C2)	$\frac{3^{47}}{2^9} - \frac{3^{319}}{2^9} + \frac{3^{313}}{2^9} - \frac{3^{353}}{2^9} + \frac{3^5}{2^7} + \frac{3^3}{2^8}$	$\left\{ \begin{array}{l} \oplus \frac{3^{47}}{2^9} + \frac{3^{313}}{2^9} + \frac{3^5}{2^7} + \frac{3^3}{2^8} = \frac{3^3}{2^9} (21+13+36+2) = \frac{3^3}{2^9} 72 \\ \ominus = \frac{3^3}{2^9} (19+53) = \frac{3^3}{2^9} 72 \end{array} \right\} = 0$
(D1)	$-\frac{3^5}{2^8} - \frac{3^4}{2^6} + \frac{3^4}{2^5} - \frac{3^4}{2^8}$	$\left\{ \begin{array}{l} \oplus = \frac{3^4}{2^5} \\ \ominus \frac{3^5}{2^8} + \frac{3^4}{2^6} + \frac{3^4}{2^8} = \frac{3^4}{2^8} (3+4+1) = \frac{3^4}{2^5} \end{array} \right\} = 0$
(D2)	$-\frac{3^4}{2^5} + \frac{3^5}{2^7} - \frac{3^3}{2^3} + \frac{3^{359}}{2^7} - \frac{3^5}{2^5} - \frac{3^3}{2^5}$	$\left\{ \begin{array}{l} \oplus = \frac{3^3}{2^7} \overbrace{(9+59)}^{68=2^2 \cdot 17} = \frac{3^3 \cdot 17}{2^5} \\ \ominus \frac{3^4}{2^5} + \frac{3^3}{2^3} + \frac{3^5}{2^5} + \frac{3^3}{2^5} = \frac{3^3}{2^5} (3+4+9+1) = \frac{3^3 \cdot 17}{2^5} \end{array} \right\} = 0$
(D3)	$-\frac{3^3}{2^3} + \frac{3 \cdot 67}{2^6} - \frac{3^3}{2^4} + \frac{3^{317}}{2^6} - \frac{3^4}{2^4} - \frac{3}{2^4}$	$\left\{ \begin{array}{l} \oplus = \frac{3}{2^6} \overbrace{(67+153)}^{220=2^2 \cdot 5 \cdot 11} = \frac{3 \cdot 5 \cdot 11}{2^4} \\ \ominus \frac{3^3}{2^3} + \frac{3^3}{2^4} + \frac{3^4}{2^4} + \frac{3}{2^4} = \frac{3}{2^4} \overbrace{(18+9+27+1)}^{55=5 \cdot 11} = \frac{3 \cdot 5 \cdot 11}{2^4} \end{array} \right\} = 0$
(E1)	$\frac{3^4}{2^3} + \frac{3^3}{2} - 3^3 + \frac{3^3}{2^3}$	$\left\{ \begin{array}{l} \oplus \frac{3^4}{2^3} + \frac{3^3}{2} + \frac{3^3}{2^3} = \frac{3^3}{2^3} (3+4+1) = 3^3 \\ \ominus = 3^3 \end{array} \right\} = 0$
(E2)	$-\frac{3^4}{2^7} + \frac{3^3}{2^3} - \frac{3^{419}}{2^7} + \frac{3^5}{2^5} + \frac{3^3}{2^4}$	$\left\{ \begin{array}{l} \oplus \frac{3^3}{2^3} + \frac{3^5}{2^5} + \frac{3^3}{2^4} = \frac{3^3}{2^5} (4+9+2) = \frac{3^4 \cdot 5}{2^5} \\ \ominus = \frac{3^4}{2^7} (1+19) = \frac{3^4 \cdot 5}{2^5} \end{array} \right\} = 0$
(E3)	$\frac{3^4}{2^3} - \frac{3^{35}}{2^4} + \frac{3^{25}}{2^2} - \frac{3^6}{2^4} + \frac{3^5}{2^3} + \frac{3^2}{2^2}$	$\left\{ \begin{array}{l} \oplus \frac{3^4}{2^3} + \frac{3^{25}}{2^2} + \frac{3^5}{2^3} + \frac{3^2}{2^2} = \frac{3^2}{2^3} \overbrace{(9+10+27+2)}^{48=2^4 \cdot 3} = 2 \cdot 3^3 \\ \ominus = \frac{3^3}{2^4} \overbrace{(5+27)}^{55=5 \cdot 11} = 2 \cdot 3^2 \end{array} \right\} = 0$
(E4)	$\frac{3^4}{2^4} - \frac{3 \cdot 13}{2^3} + \frac{3}{2} - \frac{3^3}{2^2} + \frac{3^4}{2^4}$	$\left\{ \begin{array}{l} \oplus \frac{3^4}{2^4} + \frac{3}{2} + \frac{3^4}{2^4} = \frac{3}{2^4} \overbrace{(27+8+27)}^{32=2^5} = \frac{3 \cdot 31}{2^3} \\ \ominus = \frac{3}{2^3} (13+18) = \frac{3 \cdot 31}{2^3} \end{array} \right\} = 0$
(F1)	$-3^3 + \frac{3^4}{2} - \frac{3^3}{2}$	$= \frac{3^3}{2} (-2+3-1) = 0$
(F2)	$-\frac{3^4}{2} - 3^{25} + \frac{3^{37}}{2} - 3^2$	$\left\{ \begin{array}{l} \oplus = \frac{3^7}{2} \\ \ominus = \frac{3^2}{2} (9+10+2) = \frac{3^7}{2} \end{array} \right\} = 0$
(F3)	$\frac{3^4}{2^4} - 2 \cdot 3^2 + \frac{3^{372}}{2^4} - \frac{3^5}{2^2} - 3^2$	$\left\{ \begin{array}{l} \oplus \frac{3^4}{2^4} + \frac{3^{372}}{2^4} = \frac{3^3}{2^4} \overbrace{(3+49)}^{52=2^2 \cdot 13} = \frac{3^3 \cdot 13}{2^2} \\ \ominus 2 \cdot 3^2 + \frac{3^5}{2^2} + 3^2 = \frac{3^2}{2^2} (8+27+4) = \frac{3^3 \cdot 13}{2^2} \end{array} \right\} = 0$

(F4)	$-2 \cdot 3^2 + \frac{3 \cdot 11}{2} - 2^2 3 + 2 \cdot 3^3 - \frac{3^4}{2}$	$\left\{ \begin{array}{l} \oplus \frac{3}{2}(11 + 36) = \frac{3 \cdot 47}{2} \\ \ominus \frac{3}{2}(12 + 8 + 27) = \frac{3 \cdot 47}{2} \end{array} \right\} = 0$
(F5)	$-3 + 3$	$= 0$
(G1)	$2 \cdot 3^3 + 3^3$	$= 3^4$
(G2)	$2^4 3^2 - 2 \cdot 3^3 5 + 2^3 3^2$	$= 2 \cdot 3^2(8 - 15 + 4) = -2 \cdot 3^3$
(G3)	$-\frac{3^4}{2} + \frac{3^4}{2} + 2^2 3$	$= 2^2 3$
(G4)	$2^3 3^2 + 2^4 3 - 2^2 3^3$	$= 2^2 3(6 + 4 - 9) = 2^2 3$
(G5)	$-\frac{3^3}{2} + 2^3 3 - 3^3 5 + \frac{3^5}{2}$	$\left\{ \begin{array}{l} \oplus \frac{3}{2}(16 + 81) = \frac{3 \cdot 97}{2} \\ \ominus \frac{3^3}{2}(1 + 10) = \frac{3 \cdot 99}{2} \end{array} \right\} = 0$
(G6)	$2^2 3 - 2^2 3$	$= 0$
(H1)	$-2^5 3^2 - 2^4 3^2$	$= -2^4 3^2(2 + 1) = -2^4 3^3$
(H2)	$2 \cdot 3^4 - 2^4 3^2$	$= 2 \cdot 3^2(9 - 8) = 2 \cdot 3^2$
(H3)	$-2^6 3 + 2^4 3^3$	$= 2^4 3(-4 + 9) = 2^4 3 \cdot 5$
(H4)	$2^2 3^3 - 2 \cdot 3^4$	$= 2 \cdot 3^3(2 - 3) = -2 \cdot 3^3$
(H5)	$-2^4 3$	$= -2^4 3$
(H6)	$2^2 3$	$= 2^2 3$
(I1)	$2^6 3^2$	$= 2^6 3^2$
(I2)	$2^7 3$	$= 2^7 3$
(I3)	$-2^4 3^3$	$= -2^4 3^3$
(J)	$2^8 3$	$= 2^8 3$