

4 次方程式の解の公式で得られる解と係数の関係

4 次方程式 $ax^4 + bx^3 + cx^2 + dx + e = 0$ (a, b, c, d, e は実数定数、 $a \neq 0$) の解 (x_1, x_2, x_3, x_4) と係数 (a, b, c, d, e) の関係は、

$$x_1 + x_2 + x_3 + x_4 = \sum_{n=1}^4 x_n = -\frac{b}{a} \tag{1}$$

$$x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4 = \frac{c}{a} \tag{2}$$

$$x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 = -\frac{d}{a} \tag{3}$$

$$x_1 x_2 x_3 x_4 = \prod_{n=1}^4 x_n = \frac{e}{a} \tag{4}$$

四次方程式の解の公式の結果で、この関係が成り立つことを以下示す。四次方程式の解の公式は、各解を個別に記述すると

$(k = 1)$	$i = 1$	$i = 2$	$i = 3$
L_i	$\sqrt{R + 6aP^{\frac{1}{3}} + 6aQ^{\frac{1}{3}}}$	$\sqrt{R + 6aP^{\frac{1}{3}} + 6a\omega^2 Q^{\frac{1}{3}}}$	$\sqrt{R + 6aP^{\frac{1}{3}} + 6a\omega Q^{\frac{1}{3}}}$
M_i	$\sqrt{R + 6a\omega P^{\frac{1}{3}} + 6a\omega^2 Q^{\frac{1}{3}}}$	$\sqrt{R + 6a\omega P^{\frac{1}{3}} + 6a\omega Q^{\frac{1}{3}}}$	$\sqrt{R + 6a\omega P^{\frac{1}{3}} + 6aQ^{\frac{1}{3}}}$
N_i	$\sqrt{R + 6a\omega^2 P^{\frac{1}{3}} + 6a\omega Q^{\frac{1}{3}}}$	$\sqrt{R + 6a\omega^2 P^{\frac{1}{3}} + 6aQ^{\frac{1}{3}}}$	$\sqrt{R + 6a\omega^2 P^{\frac{1}{3}} + 6a\omega^2 Q^{\frac{1}{3}}}$
x_1	$-\frac{b}{4a} + \frac{1}{12a} (L_1 + M_1 + N_1)$	$-\frac{b}{4a} + \frac{1}{12a} (L_2 + M_2 + N_2)$	$-\frac{b}{4a} + \frac{1}{12a} (L_3 + M_3 + N_3)$
x_2	$-\frac{b}{4a} + \frac{1}{12a} (L_1 - M_1 - N_1)$	$-\frac{b}{4a} + \frac{1}{12a} (L_2 - M_2 - N_2)$	$-\frac{b}{4a} + \frac{1}{12a} (L_3 - M_3 - N_3)$
x_3	$-\frac{b}{4a} + \frac{1}{12a} (-L_1 + M_1 - N_1)$	$-\frac{b}{4a} + \frac{1}{12a} (-L_2 + M_2 - N_2)$	$-\frac{b}{4a} + \frac{1}{12a} (-L_3 + M_3 - N_3)$
x_4	$-\frac{b}{4a} + \frac{1}{12a} (-L_1 - M_1 + N_1)$	$-\frac{b}{4a} + \frac{1}{12a} (-L_2 - M_2 + N_2)$	$-\frac{b}{4a} + \frac{1}{12a} (-L_3 - M_3 + N_3)$
条件 1	$P^{\frac{1}{3}} Q^{\frac{1}{3}} = 4(12ae - 3bd + c^2)$	$P^{\frac{1}{3}} Q^{\frac{1}{3}} = 4\omega(12ae - 3bd + c^2)$	$P^{\frac{1}{3}} Q^{\frac{1}{3}} = 4\omega^2(12ae - 3bd + c^2)$
条件 2	$L_i M_i N_i = 27(-8a^2 d + 4abc - b^3)$		

$(k = 2)$	$i = 1$	$i = 2$	$i = 3$
L_i	$\sqrt{R + 6aP^{\frac{1}{3}} + 6aQ^{\frac{1}{3}}}$	$\sqrt{R + 6aP^{\frac{1}{3}} + 6a\omega^2Q^{\frac{1}{3}}}$	$\sqrt{R + 6aP^{\frac{1}{3}} + 6a\omega Q^{\frac{1}{3}}}$
M_i	$\sqrt{R + 6a\omega P^{\frac{1}{3}} + 6a\omega^2Q^{\frac{1}{3}}}$	$\sqrt{R + 6a\omega P^{\frac{1}{3}} + 6a\omega Q^{\frac{1}{3}}}$	$\sqrt{R + 6a\omega P^{\frac{1}{3}} + 6aQ^{\frac{1}{3}}}$
N_i	$\sqrt{R + 6a\omega^2P^{\frac{1}{3}} + 6a\omega Q^{\frac{1}{3}}}$	$\sqrt{R + 6a\omega^2P^{\frac{1}{3}} + 6aQ^{\frac{1}{3}}}$	$\sqrt{R + 6a\omega^2P^{\frac{1}{3}} + 6a\omega^2Q^{\frac{1}{3}}}$
x_1	$-\frac{b}{4a} + \frac{1}{12a}(L_1 + M_1 - N_1)$	$-\frac{b}{4a} + \frac{1}{12a}(L_2 + M_2 - N_2)$	$-\frac{b}{4a} + \frac{1}{12a}(L_3 + M_3 - N_3)$
x_2	$-\frac{b}{4a} + \frac{1}{12a}(L_1 - M_1 + N_1)$	$-\frac{b}{4a} + \frac{1}{12a}(L_2 - M_2 + N_2)$	$-\frac{b}{4a} + \frac{1}{12a}(L_3 - M_3 + N_3)$
x_3	$-\frac{b}{4a} + \frac{1}{12a}(-L_1 + M_1 + N_1)$	$-\frac{b}{4a} + \frac{1}{12a}(-L_2 + M_2 + N_2)$	$-\frac{b}{4a} + \frac{1}{12a}(-L_3 + M_3 + N_3)$
x_4	$-\frac{b}{4a} + \frac{1}{12a}(-L_1 - M_1 - N_1)$	$-\frac{b}{4a} + \frac{1}{12a}(-L_2 - M_2 - N_2)$	$-\frac{b}{4a} + \frac{1}{12a}(-L_3 - M_3 - N_3)$
条件 1	$P^{\frac{1}{3}}Q^{\frac{1}{3}} = -4(12ae - 3bd + c^2)$	$P^{\frac{1}{3}}Q^{\frac{1}{3}} = -4\omega(12ae - 3bd + c^2)$	$P^{\frac{1}{3}}Q^{\frac{1}{3}} = -4\omega^2(12ae - 3bd + c^2)$
条件 2	$L_iM_iN_i = 27(8a^2d - 4abc + b^3)$		

式 (1) の証明

【 $k = 1, i = 1$ の場合】

$$\begin{aligned} \sum_{n=1}^4 x_n &= -\frac{b}{4a} + \frac{1}{12a}(L_1 + M_1 + N_1) - \frac{b}{4a} + \frac{1}{12a}(L_1 - M_1 - N_1) \\ &\quad - \frac{b}{4a} + \frac{1}{12a}(-L_1 + M_1 - N_1) - \frac{b}{4a} + \frac{1}{12a}(-L_1 - M_1 + N_1) \\ &= -\frac{b}{a} \end{aligned}$$

$\left. \begin{array}{l} \text{【}k = 1, i = 2 \text{ の場合】} \\ \text{【}k = 1, i = 3 \text{ の場合】} \\ \text{【}k = 2, i = 1 \text{ の場合】} \\ \text{【}k = 2, i = 2 \text{ の場合】} \\ \text{【}k = 2, i = 3 \text{ の場合】} \end{array} \right\} \text{【}k = 1, i = 1 \text{】 の場合と同様 } L_i, M_i, N_i$
 の項目が 0 になることが容易に確認できる

式 (2) の証明

【 $k = 1, i = 1$ の場合】

$$\begin{aligned} &(12a)^2(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) \\ &= \quad (-3b + L_1 + M_1 + N_1)(-3b + L_1 - M_1 - N_1) \quad \text{(A)} \\ &\quad + (-3b + L_1 + M_1 + N_1)(-3b - L_1 + M_1 - N_1) \quad \text{(B)} \\ &\quad + (-3b + L_1 + M_1 + N_1)(-3b - L_1 - M_1 + N_1) \quad \text{(C)} \\ &\quad + (-3b + L_1 - M_1 - N_1)(-3b - L_1 + M_1 - N_1) \quad \text{(D)} \\ &\quad + (-3b + L_1 - M_1 - N_1)(-3b - L_1 - M_1 + N_1) \quad \text{(E)} \\ &\quad + (-3b - L_1 + M_1 - N_1)(-3b - L_1 - M_1 + N_1) \quad \text{(F)} \end{aligned}$$

$$\begin{aligned}
(A) &= 9b^2 - 3bL_1 + 3bM_1 + 3bN_1 - 3bL_1 + L_1^2 - L_1M_1 - L_1N_1 \\
&\quad - 3bM_1 + L_1M_1 - M_1^2 - M_1N_1 - 3bN_1 + L_1N_1 - M_1N_1 - N_1^2 \\
&= 9b^2 - 6bL_1 + L_1^2 - M_1^2 - N_1^2 - 2M_1N_1 \\
(B) &= 9b^2 + 3bL_1 - 3bM_1 + 3bN_1 - 3bL_1 - L_1^2 + L_1M_1 - L_1N_1 \\
&\quad - 3bM_1 - L_1M_1 + M_1^2 - M_1N_1 - 3bN_1 - L_1N_1 + M_1N_1 - N_1^2 \\
&= 9b^2 - 6bM_1 - L_1^2 + M_1^2 - N_1^2 - 2L_1N_1 \\
(C) &= 9b^2 + 3bL_1 + 3bM_1 - 3bN_1 - 3bL_1 - L_1^2 - L_1M_1 + L_1N_1 \\
&\quad - 3bM_1 - L_1M_1 - M_1^2 + M_1N_1 - 3bN_1 - L_1N_1 - M_1N_1 + N_1^2 \\
&= 9b^2 - 6bN_1 - L_1^2 - M_1^2 + N_1^2 - 2L_1M_1 \\
(D) &= 9b^2 + 3bL_1 - 3bM_1 + 3bN_1 - 3bL_1 - L_1^2 + L_1M_1 - L_1N_1 \\
&\quad + 3bM_1 + L_1M_1 - M_1^2 + M_1N_1 + 3bN_1 + L_1N_1 - M_1N_1 + N_1^2 \\
&= 9b^2 + 6bN_1 - L_1^2 - M_1^2 + N_1^2 + 2L_1M_1 \\
(E) &= 9b^2 + 3bL_1 + 3bM_1 - 3bN_1 - 3bL_1 - L_1^2 - L_1M_1 + L_1N_1 \\
&\quad + 3bM_1 + L_1M_1 + M_1^2 - M_1N_1 + 3bN_1 + L_1N_1 + M_1N_1 - N_1^2 \\
&= 9b^2 + 6bM_1 - L_1^2 + M_1^2 - N_1^2 + 2L_1N_1 \\
(F) &= 9b^2 + 3bL_1 + 3bM_1 - 3bN_1 + 3bL_1 + L_1^2 + L_1M_1 - L_1N_1 \\
&\quad - 3bM_1 - L_1M_1 - M_1^2 + M_1N_1 + 3bN_1 + L_1N_1 + M_1N_1 - N_1^2 \\
&= 9b^2 + 6bL_1 + L_1^2 - M_1^2 - N_1^2 + 2M_1N_1
\end{aligned}$$

$$\begin{aligned}
(12a)^2(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) &= \\
&9b^2 - 6bL_1 + L_1^2 - M_1^2 - N_1^2 - 2M_1N_1 \\
&+ 9b^2 - 6bM_1 - L_1^2 + M_1^2 - N_1^2 - 2L_1N_1 \\
&+ 9b^2 - 6bN_1 - L_1^2 - M_1^2 + N_1^2 - 2L_1M_1 \\
&+ 9b^2 + 6bN_1 - L_1^2 - M_1^2 + N_1^2 + 2L_1M_1 \\
&+ 9b^2 + 6bM_1 - L_1^2 + M_1^2 - N_1^2 + 2L_1N_1 \\
&+ 9b^2 + 6bL_1 + L_1^2 - M_1^2 - N_1^2 + 2M_1N_1 \\
&= 54b^2 - 2L_1^2 - 2M_1^2 - 2N_1^2
\end{aligned}$$

$$\begin{aligned}
L_1^2 + M_1^2 + N_1^2 &= R + 6aP^{\frac{1}{3}} + 6aQ^{\frac{1}{3}} + R + 6a\omega P^{\frac{1}{3}} + 6a\omega^2 Q^{\frac{1}{3}} + R + 6a\omega^2 P^{\frac{1}{3}} + 6a\omega Q^{\frac{1}{3}} \\
&= 3R + 6a(1 + \omega + \omega^2)P^{\frac{1}{3}} + 6a(1 + \omega^2 + \omega)Q^{\frac{1}{3}} \\
&= 3(-24ac + 9b^2) \qquad \because 1 + \omega + \omega^2 = 0 \\
&= -2^3 3^2 ac + 3^3 b^2
\end{aligned} \tag{5}$$

$$\begin{aligned}
(12a)^2(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) &= \\
&= 54b^2 - 2(-2^3 3^2 ac + 3^3 b^2) \\
&= 2^4 3^2 ac
\end{aligned}$$

$$\therefore x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = \frac{2^4 3^2 ac}{(2^2 3a)^2} = \frac{c}{a}$$

【 $k = 1, i = 2$ の場合】、【 $k = 1, i = 3$ の場合】、【 $k = 2, i = 1$ の場合】、【 $k = 2, i = 2$ の場合】、【 $k = 2, i = 3$ の場合】も同様に確認すればよい（省略）。

式 (3) の証明

【 $k = 1, i = 1$ の場合】

$$\begin{aligned}
& (12a)^3(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) \\
&= (12a)^3\{x_1x_2(x_3 + x_4) + (x_1 + x_2)x_3x_4\} \\
&= (-3b + L_1 + M_1 + N_1)(-3b + L_1 - M_1 - N_1)(-3b - L_1 + M_1 - N_1 - 3b - L_1 - M_1 + N_1) \\
&\quad + (-3b + L_1 + M_1 + N_1 - 3b + L_1 - M_1 - N_1)(-3b - L_1 + M_1 - N_1)(-3b - L_1 - M_1 + N_1) \\
&= (-3b + L_1 + M_1 + N_1)(-3b + L_1 - M_1 - N_1)(-6b - 2L_1) \\
&\quad + (-6b + 2L_1)(-3b - L_1 + M_1 - N_1)(-3b - L_1 - M_1 + N_1) \\
&= (-6b - 2L_1)(9b^2 - 3bL_1 + 3bM_1 + 3bN_1 - 3bL_1 + L_1^2 - L_1M_1 - L_1N_1 \\
&\quad - 3bM_1 + L_1M_1 - M_1^2 - M_1N_1 - 3bN_1 + L_1N_1 - M_1N_1 - N_1^2) \\
&\quad + (-6b + 2L_1)(9b^2 + 3bL_1 + 3bM_1 - 3bN_1 + 3bL_1 + L_1^2 + L_1M_1 - L_1N_1 \\
&\quad - 3bM_1 - L_1M_1 - M_1^2 + M_1N_1 + 3bN_1 + L_1N_1 + M_1N_1 - N_1^2) \\
&= (-6b - 2L_1)(9b^2 - 6bL_1 + L_1^2 - M_1^2 - N_1^2 - 2M_1N_1) \\
&\quad + (-6b + 2L_1)(9b^2 + 6bL_1 + L_1^2 - M_1^2 - N_1^2 + 2M_1N_1) \\
&= \underbrace{-54b^3}_{(A)} + \underbrace{36b^2L_1}_{(B)} - \underbrace{6bL_1^2}_{(C)} + \underbrace{6bM_1^2}_{(D)} + \underbrace{6bN_1^2}_{(E)} + \underbrace{12bM_1N_1}_{(F)} \\
&\quad - \underbrace{18b^2L_1}_{(B)} + \underbrace{12bL_1^2}_{(C)} - \underbrace{2L_1^3}_{(G)} + \underbrace{2L_1M_1^2}_{(H)} + \underbrace{2L_1N_1^2}_{(I)} + \underbrace{4L_1M_1N_1}_{(J)} \\
&\quad - \underbrace{54b^3}_{(A)} - \underbrace{36b^2L_1}_{(B)} - \underbrace{6bL_1^2}_{(C)} + \underbrace{6bM_1^2}_{(D)} + \underbrace{6bN_1^2}_{(E)} - \underbrace{12bM_1N_1}_{(F)} \\
&\quad + \underbrace{18b^2L_1}_{(B)} + \underbrace{12bL_1^2}_{(C)} + \underbrace{2L_1^3}_{(G)} - \underbrace{2L_1M_1^2}_{(H)} - \underbrace{2L_1N_1^2}_{(I)} + \underbrace{4L_1M_1N_1}_{(J)} \\
&= -108b^3 + 12bL_1^2 + 12bM_1^2 + 12bN_1^2 + 8L_1M_1N_1 \quad \because (B) = (F) = (G) = (H) = (I) = 0 \\
&= -108b^3 + 12b(L_1^2 + M_1^2 + N_1^2) + 8L_1M_1N_1
\end{aligned}$$

ここで式 (5) と $L_1M_1N_1$ の条件式と合わせて代入すると、

$$\begin{aligned}
& (12a)^3(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) \\
&= -2^23^3b^3 + 2^23b(-2^33^2ac + 3^3b^2) + 2^33^3(-2^3a^2d + 2^2abc - b^3) \\
&= -2^23^3b^3 - 2^53^3abc + 2^23^4b^3 - 2^63^3a^2d + 2^53^3abc - 2^33^3b^3 \\
&= -2^63^3a^2d + (-2^53^3 + 2^53^3)abc + \frac{(-2^23^3 + 2^23^4 - 2^33^3)b^3}{=2^23^3(-1+3-2)=0} \\
&= -2^63^3a^2d
\end{aligned}$$

$$\therefore x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = -\frac{2^63^3a^2d}{(2^23a)^3} = -\frac{d}{a}$$

【 $k = 1, i = 2$ の場合】、【 $k = 1, i = 3$ の場合】、【 $k = 2, i = 1$ の場合】、【 $k = 2, i = 2$ の場合】、【 $k = 2, i = 3$ の場合】も同様に確認すればよい (省略)。

式 (4) の証明

【 $k = 1, i = 1$ の場合】

$$\begin{aligned}
(12a)^4 \prod_{n=1}^4 x_n &= (-3b + L_1 + M_1 + N_1)(-3b + L_1 - M_1 - N_1)(-3b - L_1 + M_1 - N_1)(-3b - L_1 - M_1 + N_1) \\
&= (9b^2 - 3bL_1 + 3bM_1 + 3bN_1 - 3bL_1 + L_1^2 - L_1M_1 - L_1N_1 \\
&\quad - 3bM_1 + L_1M_1 - M_1^2 - M_1N_1 - 3bN_1 + L_1N_1 - M_1N_1 - N_1^2) \\
&\quad \times (9b^2 + 3bL_1 + 3bM_1 - 3bN_1 + 3bL_1 + L_1^2 + L_1M_1 - L_1N_1 \\
&\quad - 3bM_1 - L_1M_1 - M_1^2 + M_1N_1 + 3bN_1 + L_1N_1 + M_1N_1 - N_1^2) \\
&= (9b^2 - 6bL_1 + L_1^2 - M_1^2 - N_1^2 - 2M_1N_1)(9b^2 + 6bL_1 + L_1^2 - M_1^2 - N_1^2 + 2M_1N_1) \\
&= \underset{(A)}{81b^4} + \underset{(B)}{54b^3L_1} + \underset{(C)}{9b^2L_1^2} - \underset{(D)}{9b^2M_1^2} - \underset{(E)}{9b^2N_1^2} + \underset{(F)}{18b^2M_1N_1} \\
&\quad - \underset{(B)}{54b^3L_1} - \underset{(C)}{36b^2L_1^2} - \underset{(G)}{6bL_1^3} + \underset{(H)}{6bL_1M_1^2} + \underset{(I)}{6bL_1N_1^2} - \underset{(J)}{12bL_1M_1N_1} \\
&\quad + \underset{(C)}{9b^2L_1^2} + \underset{(G)}{6bL_1^3} + \underset{(K)}{L_1^4} - \underset{(N)}{L_1^2M_1^2} - \underset{(O)}{L_1^2N_1^2} + \underset{(Q)}{2L_1^2M_1N_1} \\
&\quad - \underset{(D)}{9b^2M_1^2} - \underset{(H)}{6bL_1M_1^2} - \underset{(N)}{L_1^2M_1^2} + \underset{(L)}{M_1^4} + \underset{(P)}{M_1^2N_1^2} + \underset{(R)}{2M_1^3N_1} \\
&\quad - \underset{(E)}{9b^2N_1^2} - \underset{(I)}{6bL_1N_1^2} - \underset{(O)}{L_1^2N_1^2} + \underset{(P)}{M_1^2N_1^2} + \underset{(M)}{N_1^4} - \underset{(S)}{2M_1N_1^3} \\
&\quad - \underset{(F)}{18b^2M_1N_1} - \underset{(J)}{12bL_1M_1N_1} - \underset{(Q)}{2L_1^2M_1N_1} + \underset{(R)}{2M_1^3N_1} + \underset{(S)}{2M_1N_1^3} - \underset{(P)}{4M_1^2N_1^2} \\
&= \underset{(A)}{81b^4} - \underset{(C)}{18b^2L_1^2} - \underset{(D)}{18b^2M_1^2} - \underset{(E)}{18b^2N_1^2} - \underset{(J)}{24bL_1M_1N_1} + \underset{(L)}{L_1^4} + \underset{(M)}{M_1^4} + \underset{(N)}{N_1^4} - \underset{(O)}{2L_1^2M_1^2} - \underset{(O)}{2L_1^2N_1^2} - \underset{(P)}{2M_1^2N_1^2} \\
&\because (B) = (F) = (G) = (H) = (I) = (Q) = (R) = (S) = 0 \\
&= 81b^4 - 18b^2(L_1^2 + M_1^2 + N_1^2) - 24bL_1M_1N_1 + L_1^4 + M_1^4 + N_1^4 - 2(L_1^2M_1^2 + L_1^2N_1^2 + M_1^2N_1^2) \\
&= 81b^4 - 18b^2(L_1^2 + M_1^2 + N_1^2) - 24bL_1M_1N_1 + (L_1^2 + M_1^2 + N_1^2)^2 - 4(L_1^2M_1^2 + L_1^2N_1^2 + M_1^2N_1^2)
\end{aligned}$$

ここで、

$$\begin{aligned}
L_1^2M_1^2 &= (R + 6aP^{\frac{1}{3}} + 6aQ^{\frac{1}{3}})(R + 6a\omega P^{\frac{1}{3}} + 6a\omega^2Q^{\frac{1}{3}}) \\
&= R^2 + 6Ra\omega P^{\frac{1}{3}} + 6Ra\omega^2Q^{\frac{1}{3}} + 6RaP^{\frac{1}{3}} + 36a^2\omega P^{\frac{2}{3}} + 36a^2\omega^2P^{\frac{1}{3}}Q^{\frac{1}{3}} \\
&\quad + 6RaQ^{\frac{1}{3}} + 36a^2\omega P^{\frac{1}{3}}Q^{\frac{1}{3}} + 36a^2\omega^2Q^{\frac{2}{3}} \\
&= R^2 - 6Ra\omega^2P^{\frac{1}{3}} - 6Ra\omega Q^{\frac{1}{3}} - 36a^2P^{\frac{1}{3}}Q^{\frac{1}{3}} + 36a^2\omega P^{\frac{2}{3}} + 36a^2\omega^2Q^{\frac{2}{3}} \\
L_1^2N_1^2 &= (R + 6aP^{\frac{1}{3}} + 6aQ^{\frac{1}{3}})(R + 6a\omega^2P^{\frac{1}{3}} + 6a\omega Q^{\frac{1}{3}}) \\
&= R^2 + 6Ra\omega^2P^{\frac{1}{3}} + 6Ra\omega Q^{\frac{1}{3}} + 6RaP^{\frac{1}{3}} + 36a^2\omega^2P^{\frac{2}{3}} + 36a^2\omega P^{\frac{1}{3}}Q^{\frac{1}{3}} \\
&\quad + 6RaQ^{\frac{1}{3}} + 36a^2\omega^2P^{\frac{1}{3}}Q^{\frac{1}{3}} + 36a^2\omega Q^{\frac{2}{3}} \\
&= R^2 - 6Ra\omega P^{\frac{1}{3}} - 6Ra\omega^2Q^{\frac{1}{3}} - 36a^2P^{\frac{1}{3}}Q^{\frac{1}{3}} + 36a^2P^{\frac{2}{3}} + 36a^2Q^{\frac{2}{3}}
\end{aligned}$$

$$\begin{aligned}
M_1^2 N_1^2 &= (R + 6a\omega P^{\frac{1}{3}} + 6a\omega^2 Q^{\frac{1}{3}})(R + 6a\omega^2 P^{\frac{1}{3}} + 6a\omega Q^{\frac{1}{3}}) \\
&= R^2 + 6Ra\omega^2 P^{\frac{1}{3}} + 6Ra\omega Q^{\frac{1}{3}} + 6Ra\omega P^{\frac{1}{3}} + 36a^2 P^{\frac{2}{3}} + 36a^2 \omega^2 P^{\frac{1}{3}} Q^{\frac{1}{3}} \\
&\quad + 6Ra\omega^2 Q^{\frac{1}{3}} + 36a^2 \omega P^{\frac{1}{3}} Q^{\frac{1}{3}} + 36a^2 Q^{\frac{2}{3}} \\
&= R^2 - 6RaP^{\frac{1}{3}} - 6RaQ^{\frac{1}{3}} - 36a^2 P^{\frac{1}{3}} Q^{\frac{1}{3}} + 36a^2 P^{\frac{2}{3}} + 36a^2 Q^{\frac{2}{3}} \\
L_1^2 M_1^2 + L_1^2 N_1^2 + M_1^2 N_1^2 &= R^2 - 6Ra\omega^2 P^{\frac{1}{3}} - 6Ra\omega Q^{\frac{1}{3}} - 36a^2 P^{\frac{1}{3}} Q^{\frac{1}{3}} + 36a^2 \omega P^{\frac{2}{3}} + 36a^2 \omega^2 Q^{\frac{2}{3}} \\
&\quad + R^2 - 6Ra\omega P^{\frac{1}{3}} - 6Ra\omega^2 Q^{\frac{1}{3}} - 36a^2 P^{\frac{1}{3}} Q^{\frac{1}{3}} + 36a^2 P^{\frac{2}{3}} + 36a^2 Q^{\frac{2}{3}} \\
&\quad + R^2 - 6RaP^{\frac{1}{3}} - 6RaQ^{\frac{1}{3}} - 36a^2 P^{\frac{1}{3}} Q^{\frac{1}{3}} + 36a^2 P^{\frac{2}{3}} + 36a^2 Q^{\frac{2}{3}} \\
&= 3R^2 - 2^2 3^3 a^2 P^{\frac{1}{3}} Q^{\frac{1}{3}} \\
&= \frac{1}{3} (3R)^2 - 2^2 3^3 a^2 P^{\frac{1}{3}} Q^{\frac{1}{3}} \\
&= \frac{1}{3} (-2^3 3^2 ac + 3^3 b^2)^2 - 2^2 3^3 a^2 \cdot 2^2 (2^2 3ae - 3bd + c^2) \\
&= \frac{1}{3} (2^6 3^4 a^2 c^2 - 2^4 3^5 ab^2 c + 3^6 b^4) - 2^6 3^4 a^3 e + 2^4 3^4 a^2 bd - 2^4 3^3 a^2 c^2 \\
&= -2^6 3^4 a^3 e + (2^6 3^3 - 2^4 3^3) a^2 c^2 + 2^4 3^4 a^2 bd - 2^4 3^4 ab^2 c + 3^5 b^4 \\
&= -2^6 3^4 a^3 e + 2^4 3^4 a^2 c^2 + 2^4 3^4 a^2 bd - 2^4 3^4 ab^2 c + 3^5 b^4
\end{aligned}$$

なので、条件2と式(5)を合わせて代入すると

$$\begin{aligned}
(12a)^4 \prod_{n=1}^4 x_n &= 3^4 b^4 - 2 \cdot 3^2 b^2 (-2^3 3^2 ac + 3^3 b^2) - 2^3 3b \cdot 3^3 (-2^3 a^2 d + 2^2 abc - b^3) + (-2^3 3^2 ac + 3^3 b^2)^2 \\
&\quad - 2^2 (-2^6 3^4 a^3 e + 2^4 3^4 a^2 c^2 + 2^4 3^4 a^2 bd - 2^4 3^4 ab^2 c + 3^5 b^4) \\
&= 3^4 b^4 + 2^4 3^4 ab^2 c - 2 \cdot 3^5 b^4 + 2^6 3^4 a^2 bd - 2^5 3^4 ab^2 c + 2^3 3^4 b^4 \\
&\quad + 2^6 3^4 a^2 c^2 - 2^4 3^5 ab^2 c + 3^6 b^4 \\
&\quad + 2^8 3^4 a^3 e - 2^6 3^4 a^2 c^2 - 2^6 3^4 a^2 bd + 2^6 3^4 ab^2 c - 2^2 3^5 b^4 \\
&= 2^8 3^4 a^3 e + (2^6 3^4 - 2^6 3^4) a^2 bd + (2^6 3^4 - 2^6 3^4) a^2 c^2 \\
&\quad + \frac{(2^4 3^4 - 2^5 3^4 - 2^4 3^5 + 2^6 3^4) ab^2 c}{=2^4 3^4 (1-2-3+4)=0} + \frac{(3^4 - 2 \cdot 3^5 + 2^3 3^4 + 3^6 - 2^2 3^5) b^4}{=3^4 (1-6+8+9-12)=0} \\
\therefore \prod_{n=1}^4 x_n &= \frac{2^8 3^4 a^3 e}{(2^2 3a)^4} = \frac{e}{a}
\end{aligned}$$

【 $k=1, i=2$ の場合】、【 $k=1, i=3$ の場合】、【 $k=2, i=1$ の場合】、【 $k=2, i=2$ の場合】、【 $k=2, i=3$ の場合】も同様に確認すればよい(省略)。

以上、四次方程式の解の公式の6つの選択肢($k=1, 2$ 及び $i=1, 2, 3$ の組合せ)のいずれを選んでも、解と係数の関係(1)、(2)、(3)、(4)が成立する。 ■