

## 3次方程式の解の公式で得られる解と係数の関係

3次方程式  $ax^3 + bx^2 + cx + d = 0$  ( $a, b, c, d$  は実数定数、 $a \neq 0$ ) の解 ( $x_1, x_2, x_3$ ) と係数 ( $a, b, c, d$ ) の関係は、

$$x_1 + x_2 + x_3 = \sum_{k=1}^3 x_k = -\frac{b}{a} \quad (1)$$

$$x_1 x_2 + x_2 x_3 + x_3 x_1 = \frac{c}{a} \quad (2)$$

$$x_1 x_2 x_3 = \prod_{k=1}^3 x_k = -\frac{d}{a} \quad (3)$$

三次方程式の解の公式の結果で、この関係が成り立つことを以下示す。三次方程式の解の公式は、各解を個別に記述すると

	$x_1$	$x_2$	$x_3$	$P^{\frac{1}{3}}Q^{\frac{1}{3}}$
$i = 1$	$\frac{1}{6a} \left( -2b + P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right)$	$\frac{1}{6a} \left( -2b + \omega P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}} \right)$	$\frac{1}{6a} \left( -2b + \omega^2 P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}} \right)$	$4b^2 - 12ac$
$i = 2$	$\frac{1}{6a} \left( -2b + P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}} \right)$	$\frac{1}{6a} \left( -2b + \omega P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}} \right)$	$\frac{1}{6a} \left( -2b + \omega^2 P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right)$	$\omega (4b^2 - 12ac)$
$i = 3$	$\frac{1}{6a} \left( -2b + P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}} \right)$	$\frac{1}{6a} \left( -2b + \omega P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right)$	$\frac{1}{6a} \left( -2b + \omega^2 P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}} \right)$	$\omega^2 (4b^2 - 12ac)$

## 式 (1) の証明

【 $i = 1$  の場合】

$$\begin{aligned} \sum_{k=1}^3 x_k &= \frac{1}{6a} \left( -2b + P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right) + \frac{1}{6a} \left( -2b + \omega P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}} \right) + \frac{1}{6a} \left( -2b + \omega^2 P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}} \right) \\ &= \frac{1}{6a} \left\{ -6b + (1 + \omega + \omega^2) P^{\frac{1}{3}} + (1 + \omega^2 + \omega) Q^{\frac{1}{3}} \right\} = -\frac{b}{a} \end{aligned}$$

【 $i = 2$  の場合】

$$\begin{aligned} \sum_{k=1}^3 x_k &= \frac{1}{6a} \left( -2b + P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}} \right) + \frac{1}{6a} \left( -2b + \omega P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}} \right) + \frac{1}{6a} \left( -2b + \omega^2 P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right) \\ &= \frac{1}{6a} \left\{ -6b + (1 + \omega + \omega^2) P^{\frac{1}{3}} + (\omega^2 + \omega + 1) Q^{\frac{1}{3}} \right\} = -\frac{b}{a} \end{aligned}$$

【 $i = 3$  の場合】

$$\begin{aligned} \sum_{k=1}^3 x_k &= \frac{1}{6a} \left( -2b + P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}} \right) + \frac{1}{6a} \left( -2b + \omega P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right) + \frac{1}{6a} \left( -2b + \omega^2 P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}} \right) \\ &= \frac{1}{6a} \left\{ -6b + (1 + \omega + \omega^2) P^{\frac{1}{3}} + (\omega + 1 + \omega^2) Q^{\frac{1}{3}} \right\} = -\frac{b}{a} \end{aligned}$$

式 (2) の証明

【 $i = 1$  の場合】

$$\begin{aligned}
x_1 x_2 + x_2 x_3 + x_3 x_1 &= \frac{1}{36a^2} \left( -2b + P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right) \left( -2b + \omega P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}} \right) \\
&\quad + \frac{1}{36a^2} \left( -2b + \omega P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}} \right) \left( -2b + \omega^2 P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}} \right) \\
&\quad + \frac{1}{36a^2} \left( -2b + \omega^2 P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}} \right) \left( -2b + P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right). \\
&= \frac{1}{36a^2} \left( 4b^2 - 2b\omega P^{\frac{1}{3}} - 2b\omega^2 Q^{\frac{1}{3}} - 2bP^{\frac{1}{3}} + \omega P^{\frac{2}{3}} + \omega^2 P^{\frac{1}{3}} Q^{\frac{1}{3}} - 2bQ^{\frac{1}{3}} + \omega P^{\frac{1}{3}} Q^{\frac{1}{3}} + \omega^2 Q^{\frac{2}{3}} \right. \\
&\quad + 4b^2 - 2b\omega^2 P^{\frac{1}{3}} - 2b\omega Q^{\frac{1}{3}} - 2b\omega P^{\frac{1}{3}} + P^{\frac{2}{3}} + \omega^2 P^{\frac{1}{3}} Q^{\frac{1}{3}} - 2b\omega^2 Q^{\frac{1}{3}} + \omega P^{\frac{1}{3}} Q^{\frac{1}{3}} + Q^{\frac{2}{3}} \\
&\quad \left. + 4b^2 - 2bP^{\frac{1}{3}} - 2bQ^{\frac{1}{3}} - 2b\omega^2 P^{\frac{1}{3}} + \omega^2 P^{\frac{2}{3}} + \omega^2 P^{\frac{1}{3}} Q^{\frac{1}{3}} - 2b\omega Q^{\frac{1}{3}} + \omega P^{\frac{1}{3}} Q^{\frac{1}{3}} + \omega Q^{\frac{2}{3}} \right) \\
&= \frac{1}{36a^2} \left\{ 12b^2 - 2b(\omega + \omega^2 + 1)P^{\frac{1}{3}} - 2b(\omega^2 + \omega + 1)Q^{\frac{1}{3}} - 2b(1 + \omega + \omega^2)P^{\frac{1}{3}} \right. \\
&\quad \left. + (\omega + 1 + \omega^2)P^{\frac{2}{3}} + 3\omega^2 P^{\frac{1}{3}} Q^{\frac{1}{3}} - 2b(1 + \omega^2 + \omega)Q^{\frac{1}{3}} + 3\omega P^{\frac{1}{3}} Q^{\frac{1}{3}} + (\omega^2 + 1 + \omega)Q^{\frac{2}{3}} \right\} \\
&= \frac{1}{36a^2} \left( 12b^2 - 3P^{\frac{1}{3}} Q^{\frac{1}{3}} \right)
\end{aligned}$$

ここで、三次方程式の解の公式の条件 ( $i = 1$  の場合の  $P^{\frac{1}{3}} Q^{\frac{1}{3}} = 4b^2 - 12ac$ ) を用いて、

$$x_1 x_2 + x_2 x_3 + x_3 x_1 = \frac{1}{36a^2} \{ 12b^2 - 3(4b^2 - 12ac) \} = \frac{c}{a}$$

【 $i = 2$  の場合】

$$\begin{aligned}
x_1 x_2 + x_2 x_3 + x_3 x_1 &= \frac{1}{36a^2} \left( -2b + P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}} \right) \left( -2b + \omega P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}} \right) \\
&\quad + \frac{1}{36a^2} \left( -2b + \omega P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}} \right) \left( -2b + \omega^2 P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right) \\
&\quad + \frac{1}{36a^2} \left( -2b + \omega^2 P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right) \left( -2b + P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}} \right). \\
&= \frac{1}{36a^2} \left( 4b^2 - 2b\omega P^{\frac{1}{3}} - 2b\omega Q^{\frac{1}{3}} - 2bP^{\frac{1}{3}} + \omega P^{\frac{2}{3}} + \omega P^{\frac{1}{3}} Q^{\frac{1}{3}} - 2b\omega^2 Q^{\frac{1}{3}} + P^{\frac{1}{3}} Q^{\frac{1}{3}} + Q^{\frac{2}{3}} \right. \\
&\quad + 4b^2 - 2b\omega^2 P^{\frac{1}{3}} - 2bQ^{\frac{1}{3}} - 2b\omega P^{\frac{1}{3}} + P^{\frac{2}{3}} + \omega P^{\frac{1}{3}} Q^{\frac{1}{3}} - 2b\omega Q^{\frac{1}{3}} + P^{\frac{1}{3}} Q^{\frac{1}{3}} + \omega Q^{\frac{2}{3}} \\
&\quad \left. + 4b^2 - 2bP^{\frac{1}{3}} - 2b\omega^2 Q^{\frac{1}{3}} - 2b\omega^2 P^{\frac{1}{3}} + \omega^2 P^{\frac{2}{3}} + \omega P^{\frac{1}{3}} Q^{\frac{1}{3}} - 2bQ^{\frac{1}{3}} + P^{\frac{1}{3}} Q^{\frac{1}{3}} + \omega^2 Q^{\frac{2}{3}} \right) \\
&= \frac{1}{36a^2} \left\{ 12b^2 - 2b(\omega + \omega^2 + 1)P^{\frac{1}{3}} - 2b(\omega + 1 + \omega^2)Q^{\frac{1}{3}} - 2b(1 + \omega + \omega^2)P^{\frac{1}{3}} \right. \\
&\quad \left. + (\omega + 1 + \omega^2)P^{\frac{2}{3}} + 3\omega P^{\frac{1}{3}} Q^{\frac{1}{3}} - 2b(\omega^2 + \omega + 1)Q^{\frac{1}{3}} + 3P^{\frac{1}{3}} Q^{\frac{1}{3}} + (1 + \omega + \omega^2)Q^{\frac{2}{3}} \right\} \\
&= \frac{1}{36a^2} \left( 12b^2 - 3\omega^2 P^{\frac{1}{3}} Q^{\frac{1}{3}} \right)
\end{aligned}$$

ここで、三次方程式の解の公式の条件 ( $i = 2$  の場合の  $P^{\frac{1}{3}} Q^{\frac{1}{3}} = \omega(4b^2 - 12ac)$ ) を用いて、

$$x_1 x_2 + x_2 x_3 + x_3 x_1 = \frac{1}{36a^2} \{ 12b^2 - 3\omega^2 \cdot \omega(4b^2 - 12ac) \} = \frac{c}{a}$$

【 $i = 3$  の場合】

$$\begin{aligned}
x_1 x_2 + x_2 x_3 + x_3 x_1 &= \frac{1}{36a^2} \left( -2b + P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}} \right) \left( -2b + \omega P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right) \\
&\quad + \frac{1}{36a^2} \left( -2b + \omega P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right) \left( -2b + \omega^2 P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}} \right) \\
&\quad + \frac{1}{36a^2} \left( -2b + \omega^2 P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}} \right) \left( -2b + P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}} \right) \cdot \\
&= \frac{1}{36a^2} \left( 4b^2 - 2b\omega P^{\frac{1}{3}} - 2bQ^{\frac{1}{3}} - 2bP^{\frac{1}{3}} + \omega P^{\frac{2}{3}} + P^{\frac{1}{3}}Q^{\frac{1}{3}} - 2b\omega Q^{\frac{1}{3}} + \omega^2 P^{\frac{1}{3}}Q^{\frac{1}{3}} + \omega Q^{\frac{2}{3}} \right. \\
&\quad + 4b^2 - 2b\omega^2 P^{\frac{1}{3}} - 2b\omega^2 Q^{\frac{1}{3}} - 2b\omega P^{\frac{1}{3}} + P^{\frac{2}{3}} + P^{\frac{1}{3}}Q^{\frac{1}{3}} - 2bQ^{\frac{1}{3}} + \omega^2 P^{\frac{1}{3}}Q^{\frac{1}{3}} + \omega^2 Q^{\frac{2}{3}} \\
&\quad \left. + 4b^2 - 2bP^{\frac{1}{3}} - 2b\omega Q^{\frac{1}{3}} - 2b\omega^2 P^{\frac{1}{3}} + \omega^2 P^{\frac{2}{3}} + P^{\frac{1}{3}}Q^{\frac{1}{3}} - 2b\omega^2 Q^{\frac{1}{3}} + \omega^2 P^{\frac{1}{3}}Q^{\frac{1}{3}} + 2Q^{\frac{2}{3}} \right) \\
&= \frac{1}{36a^2} \left\{ 12b^2 - 2b(\omega + \omega^2 + 1)P^{\frac{1}{3}} - 2b(1 + \omega^2 + \omega)Q^{\frac{1}{3}} - 2b(1 + \omega + \omega^2)P^{\frac{1}{3}} \right. \\
&\quad \left. + (\omega + 1 + \omega^2)P^{\frac{2}{3}} + 3P^{\frac{1}{3}}Q^{\frac{1}{3}} - 2b(\omega + 1 + \omega^2)Q^{\frac{1}{3}} + 3\omega^2 P^{\frac{1}{3}}Q^{\frac{1}{3}} + (\omega + \omega^2 + 1)Q^{\frac{2}{3}} \right\} \\
&= \frac{1}{36a^2} \left( 12b^2 - 3\omega P^{\frac{1}{3}}Q^{\frac{1}{3}} \right)
\end{aligned}$$

ここで、三次方程式の解の公式の条件 ( $i = 3$  の場合の  $P^{\frac{1}{3}}Q^{\frac{1}{3}} = \omega^2(4b^2 - 12ac)$ ) を用いて、

$$x_1 x_2 + x_2 x_3 + x_3 x_1 = \frac{1}{36a^2} \{ 12b^2 - 3\omega \cdot \omega^2 (4b^2 - 12ac) \} = \frac{c}{a}$$

式 (3) の証明

【 $i = 1$  の場合】

$$\begin{aligned}
(6a)^3 \prod_{k=1}^3 x_k &= \left( -2b + P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right) \left( -2b + \omega P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}} \right) \left( -2b + \omega^2 P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}} \right) \\
&= \left( -2b + P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right) \\
&\quad \times \left( 4b^2 - 2b\omega^2 P^{\frac{1}{3}} - 2b\omega Q^{\frac{1}{3}} - 2b\omega P^{\frac{1}{3}} + P^{\frac{2}{3}} + \omega^2 P^{\frac{1}{3}}Q^{\frac{1}{3}} - 2b\omega^2 Q^{\frac{1}{3}} + \omega P^{\frac{1}{3}}Q^{\frac{1}{3}} + Q^{\frac{2}{3}} \right) \\
&= \left( -2b + P^{\frac{1}{3}} + Q^{\frac{1}{3}} \right) \left( 4b^2 + 2bP^{\frac{1}{3}} + 2bQ^{\frac{1}{3}} - P^{\frac{1}{3}}Q^{\frac{1}{3}} + P^{\frac{2}{3}} + Q^{\frac{2}{3}} \right) \\
&= \underbrace{-8b^3}_{(A)} - \underbrace{4b^2 P^{\frac{1}{3}}}_{(B)} - \underbrace{4b^2 Q^{\frac{1}{3}}}_{(C)} + \underbrace{2b P^{\frac{1}{3}} Q^{\frac{1}{3}}}_{(D)} - \underbrace{2b P^{\frac{2}{3}}}_{(E)} - \underbrace{2b Q^{\frac{2}{3}}}_{(F)} \\
&\quad + \underbrace{4b^2 P^{\frac{1}{3}}}_{(B)} + \underbrace{2b P^{\frac{2}{3}}}_{(E)} + \underbrace{2b P^{\frac{1}{3}} Q^{\frac{1}{3}}}_{(D)} - \underbrace{P^{\frac{2}{3}} Q^{\frac{1}{3}}}_{(G)} + \underbrace{P}_{(I)} + \underbrace{P^{\frac{1}{3}} Q^{\frac{2}{3}}}_{(H)} \\
&\quad + \underbrace{4b^2 Q^{\frac{1}{3}}}_{(C)} + \underbrace{2b P^{\frac{1}{3}} Q^{\frac{1}{3}}}_{(D)} + \underbrace{2b Q^{\frac{2}{3}}}_{(F)} - \underbrace{P^{\frac{1}{3}} Q^{\frac{2}{3}}}_{(H)} + \underbrace{P^{\frac{2}{3}} Q^{\frac{1}{3}}}_{(G)} + \underbrace{Q}_{(J)} \\
&= \underbrace{-8b^3}_{(A)} + \underbrace{6b P^{\frac{1}{3}} Q^{\frac{1}{3}}}_{(D)} + \underbrace{P}_{(I)} + \underbrace{Q}_{(J)} \\
&\quad \therefore (B) = 0, (C) = 0, (E) = 0, (F) = 0, (G) = 0, (H) = 0
\end{aligned}$$

ここで、 $P + Q$  は解の公式より

$$P + Q = 2X = 2(-2^2 3^3 a^2 d + 2^2 3^2 abc - 2^3 b^3) = -2^3 3^3 a^2 d + 2^3 3^2 abc - 2^4 b^3 \quad (4)$$

更に、三次方程式の解の公式の条件式 ( $i = 1$  の場合 :  $P^{\frac{1}{3}}Q^{\frac{1}{3}} = 4b^2 - 12ac$ ) を用いて

$$\begin{aligned}
 (2 \cdot 3a)^3 \prod_{k=1}^3 x_k &= -8b^3 + 6b(4b^2 - 12ac) - 2^3 3^3 a^2 d + 2^3 3^2 abc - 2^4 b^3 \\
 &= -2^3 b^3 + 2^3 3b^3 - 2^3 3^2 abc - 2^3 3^3 a^2 d + 2^3 3^2 abc - 2^4 b^3 \\
 &= \underbrace{(-2^3 + 2^3 3 - 2^4)}_{=0} b^3 - \underbrace{(2^3 3^2 - 2^3 3^2)}_{=0} abc - 2^3 3^3 a^2 d \\
 \therefore \prod_{k=1}^3 x_k &= -\frac{2^3 3^3 a^2 d}{2^3 3^3 a^3} = -\frac{d}{a}
 \end{aligned}$$

【 $i = 2$  の場合】

$$\begin{aligned}
 (6a)^3 \prod_{k=1}^3 x_k &= \left(-2b + P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}}\right) \left(-2b + \omega P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}}\right) \left(-2b + \omega^2 P^{\frac{1}{3}} + Q^{\frac{1}{3}}\right) \\
 &= \left(-2b + P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}}\right) \\
 &\quad \times \left(4b^2 - 2b\omega^2 P^{\frac{1}{3}} - 2b\omega Q^{\frac{1}{3}} - 2b\omega P^{\frac{1}{3}} + P^{\frac{2}{3}} + \omega P^{\frac{1}{3}} Q^{\frac{1}{3}} - 2b\omega Q^{\frac{1}{3}} + P^{\frac{1}{3}} Q^{\frac{1}{3}} + \omega Q^{\frac{2}{3}}\right) \\
 &= \left(-2b + P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}}\right) \left(4b^2 + 2bP^{\frac{1}{3}} + 2b\omega^2 Q^{\frac{1}{3}} - \omega^2 P^{\frac{1}{3}} Q^{\frac{1}{3}} + P^{\frac{2}{3}} + \omega Q^{\frac{2}{3}}\right) \\
 &= \underbrace{-8b^3}_{(A)} - \underbrace{4b^2 P^{\frac{1}{3}}}_{(B)} - \underbrace{4b^2 \omega^2 Q^{\frac{1}{3}}}_{(C)} + \underbrace{2b\omega^2 P^{\frac{1}{3}} Q^{\frac{1}{3}}}_{(D)} - \underbrace{2bP^{\frac{2}{3}}}_{(E)} - \underbrace{2b\omega Q^{\frac{2}{3}}}_{(F)} \\
 &\quad + \underbrace{4b^2 P^{\frac{1}{3}}}_{(B)} + \underbrace{2bP^{\frac{2}{3}}}_{(E)} + \underbrace{2b\omega^2 P^{\frac{1}{3}} Q^{\frac{1}{3}}}_{(D)} - \underbrace{\omega^2 P^{\frac{2}{3}} Q^{\frac{1}{3}}}_{(G)} + \underbrace{P}_{(I)} + \underbrace{\omega P^{\frac{1}{3}} Q^{\frac{2}{3}}}_{(H)} \\
 &\quad + \underbrace{4b^2 \omega^2 Q^{\frac{1}{3}}}_{(C)} + \underbrace{2b\omega^2 P^{\frac{1}{3}} Q^{\frac{1}{3}}}_{(D)} + \underbrace{2b\omega Q^{\frac{2}{3}}}_{(F)} - \underbrace{\omega P^{\frac{1}{3}} Q^{\frac{2}{3}}}_{(H)} + \underbrace{\omega^2 P^{\frac{2}{3}} Q^{\frac{1}{3}}}_{(G)} + \underbrace{Q}_{(J)} \\
 &= \underbrace{-8b^3}_{(A)} + \underbrace{6b\omega^2 P^{\frac{1}{3}} Q^{\frac{1}{3}}}_{(D)} + \underbrace{P}_{(I)} + \underbrace{Q}_{(J)} \\
 &\quad \therefore (B) = 0, (C) = 0, (E) = 0, (F) = 0, (G) = 0, (H) = 0
 \end{aligned}$$

ここで、 $P + Q$  は  $i$  にかかわらず式 (4) が成立するので、式 (4) と三次方程式の解の公式の条件式 [ $i = 2$  の場合 :  $P^{\frac{1}{3}}Q^{\frac{1}{3}} = \omega(4b^2 - 12ac)$ ] を用いて

$$\begin{aligned}
 (2 \cdot 3a)^3 \prod_{k=1}^3 x_k &= -8b^3 + 6b\omega^2 \cdot \omega(4b^2 - 12ac) - 2^3 3^3 a^2 d + 2^3 3^2 abc - 2^4 b^3 \\
 &= -2^3 b^3 + 2^3 3b^3 - 2^3 3^2 abc - 2^3 3^3 a^2 d + 2^3 3^2 abc - 2^4 b^3 \\
 &= \underbrace{(-2^3 + 2^3 3 - 2^4)}_{=0} b^3 - \underbrace{(2^3 3^2 - 2^3 3^2)}_{=0} abc - 2^3 3^3 a^2 d \\
 \therefore \prod_{k=1}^3 x_k &= -\frac{2^3 3^3 a^2 d}{2^3 3^3 a^3} = -\frac{d}{a}
 \end{aligned}$$

【 $i = 3$  の場合】

$$\begin{aligned}
 (6a)^3 \prod_{k=1}^3 x_k &= \left(-2b + P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}}\right) \left(-2b + \omega P^{\frac{1}{3}} + Q^{\frac{1}{3}}\right) \left(-2b + \omega^2 P^{\frac{1}{3}} + \omega^2 Q^{\frac{1}{3}}\right) \\
 &= \left(-2b + P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}}\right) \\
 &\quad \times \left(4b^2 - 2b\omega^2 P^{\frac{1}{3}} - 2b\omega^2 Q^{\frac{1}{3}} - 2b\omega P^{\frac{1}{3}} + P^{\frac{2}{3}} + P^{\frac{1}{3}} Q^{\frac{1}{3}} - 2bQ^{\frac{1}{3}} + \omega^2 P^{\frac{1}{3}} Q^{\frac{1}{3}} + \omega^2 Q^{\frac{2}{3}}\right) \\
 &= \left(-2b + P^{\frac{1}{3}} + \omega Q^{\frac{1}{3}}\right) \left(4b^2 + 2bP^{\frac{1}{3}} + 2b\omega Q^{\frac{1}{3}} - \omega P^{\frac{1}{3}} Q^{\frac{1}{3}} + P^{\frac{2}{3}} + \omega^2 Q^{\frac{2}{3}}\right) \\
 &= \underbrace{-8b^3}_{(A)} - \underbrace{4b^2 P^{\frac{1}{3}}}_{(B)} - \underbrace{4b^2 \omega Q^{\frac{1}{3}}}_{(C)} + \underbrace{2b\omega P^{\frac{1}{3}} Q^{\frac{1}{3}}}_{(D)} - \underbrace{2bP^{\frac{2}{3}}}_{(E)} - \underbrace{2b\omega^2 Q^{\frac{2}{3}}}_{(F)} \\
 &\quad + \underbrace{4b^2 P^{\frac{1}{3}}}_{(B)} + \underbrace{2bP^{\frac{2}{3}}}_{(E)} + \underbrace{2b\omega P^{\frac{1}{3}} Q^{\frac{1}{3}}}_{(D)} - \underbrace{\omega P^{\frac{2}{3}} Q^{\frac{1}{3}}}_{(G)} + \underbrace{P}_{(I)} + \underbrace{\omega^2 P^{\frac{1}{3}} Q^{\frac{2}{3}}}_{(H)} \\
 &\quad + \underbrace{4b^2 \omega Q^{\frac{1}{3}}}_{(C)} + \underbrace{2b\omega P^{\frac{1}{3}} Q^{\frac{1}{3}}}_{(D)} + \underbrace{2b\omega^2 Q^{\frac{2}{3}}}_{(F)} - \underbrace{\omega^2 P^{\frac{1}{3}} Q^{\frac{2}{3}}}_{(H)} + \underbrace{\omega P^{\frac{2}{3}} Q^{\frac{1}{3}}}_{(G)} + \underbrace{Q}_{(J)} \\
 &= \underbrace{-8b^3}_{(A)} + \underbrace{6b\omega P^{\frac{1}{3}} Q^{\frac{1}{3}}}_{(D)} + \underbrace{P}_{(I)} + \underbrace{Q}_{(J)} \\
 &\quad \therefore (B) = 0, (C) = 0, (E) = 0, (F) = 0, (G) = 0, (H) = 0
 \end{aligned}$$

ここで、 $P + Q$  は  $i$  にかかわらず式 (4) が成立するので、式 (4) と三次方程式の解の公式の条件式 [ $i = 3$  の場合： $P^{\frac{1}{3}} Q^{\frac{1}{3}} = \omega^2(4b^2 - 12ac)$ ] を用いて

$$\begin{aligned}
 (2 \cdot 3a)^3 \prod_{k=1}^3 x_k &= -8b^3 + 6b\omega \cdot \omega^2(4b^2 - 12ac) - 2^3 3^3 a^2 d + 2^3 3^2 abc - 2^4 b^3 \\
 &= -2^3 b^3 + 2^3 3b^3 - 2^3 3^2 abc - 2^3 3^3 a^2 d + 2^3 3^2 abc - 2^4 b^3 \\
 &= \underbrace{(-2^3 + 2^3 3 - 2^4)}_{=0} b^3 - \underbrace{(2^3 3^2 - 2^3 3^2)}_{=0} abc - 2^3 3^3 a^2 d \\
 \therefore \prod_{k=1}^3 x_k &= -\frac{2^3 3^3 a^2 d}{2^3 3^3 a^3} = -\frac{d}{a}
 \end{aligned}$$

以上、三次方程式の解の公式の3つの選択肢 ( $i = 1, 2, 3$ ) のいずれを選んでも、解と係数の関係 (1)、(2)、(3) が成立する。 ■